Lattice Quantum Chromo Dynamics

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Introduction: Why & How

Scalar Fields

Quarks on lattice

Bringing in Interactions

Continuum Limit

Introduction : Quantum Chromo Dynamics (QCD)

- Gauge theory of interactions of quarks & gluons. Similar in structure to theory of electrons & photons (QED).

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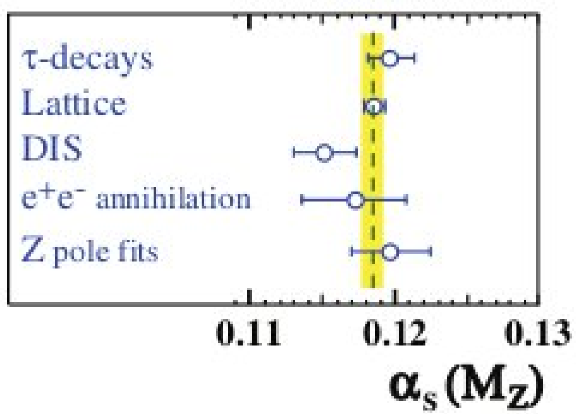
- Many more “photons” (Eight) which carry colour charge & hence interact amongst themselves.

- Asymptotic Freedom : Coupling α_s small for large momentum transfer. (Nobel Prize 2004).

- Tested extensively in many experiments: DIS, Z-pole, $R_{e^+e^-}$,...



From Particle Data Group 2012

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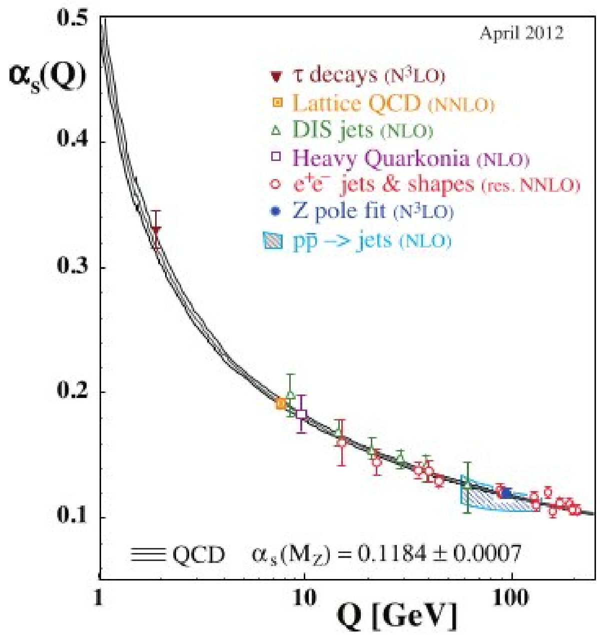

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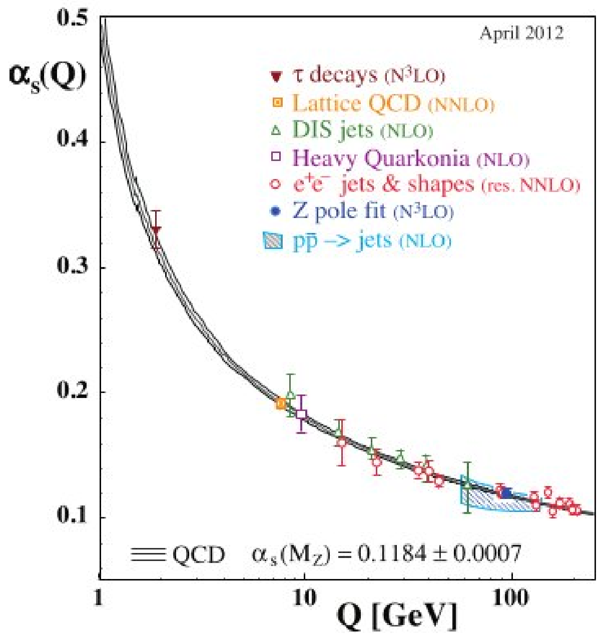

- *Unlike QED*, the coupling can be very large.

- Much richer structure : Quark Confinement, Dynamical Symmetry Breaking..

- $M_{Proton} \gg (2m_u + m_d)$, by a factor of 100 $\rightarrow$ Understanding it is knowing where the Visible mass of Universe comes from.

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- For (N_f) massless particles, A or B do **not** change into each other: Chiral Symmetry $(SU(N_f) \times SU(N_f))$.

- Interactions can break the chiral symmetry dynamically, leading to effective masses for these particles.
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- New States at High Temperatures/Density expected on basis of models.
- Quark-Gluon Plasma is such a phase. It presumably filled our Universe a few microseconds after the Big Bang & can be produced in Relativistic Heavy Ion Collisions.
- Much richer structure in QCD : Quark Confinement, Dynamical Symmetry Breaking.. Lattice QCD should shed light on this all.

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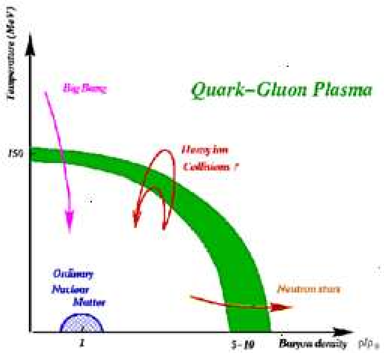

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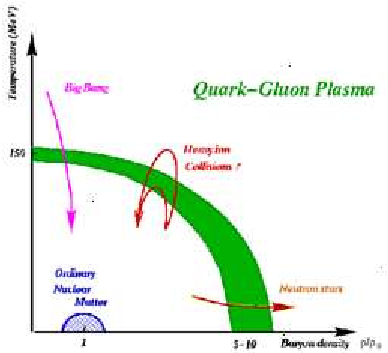

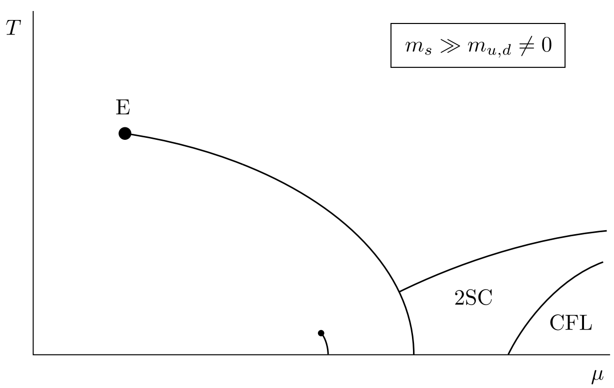

From Columbia Physics Webpage & Rajagopal-Wilczek Review

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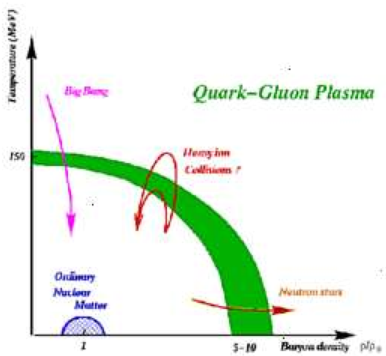

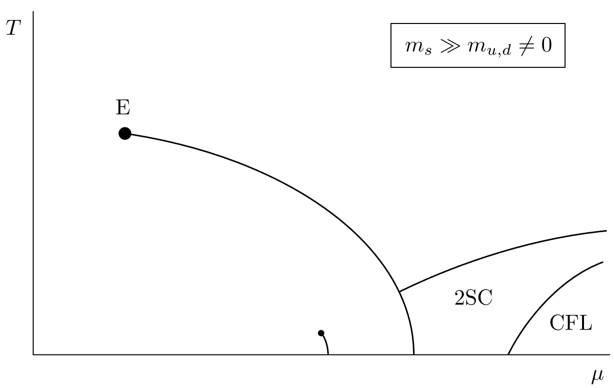

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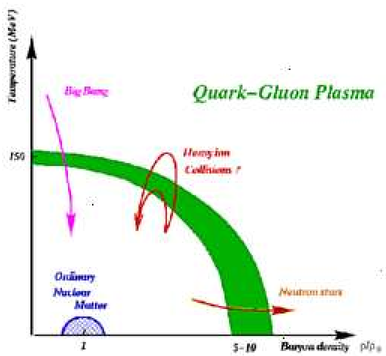

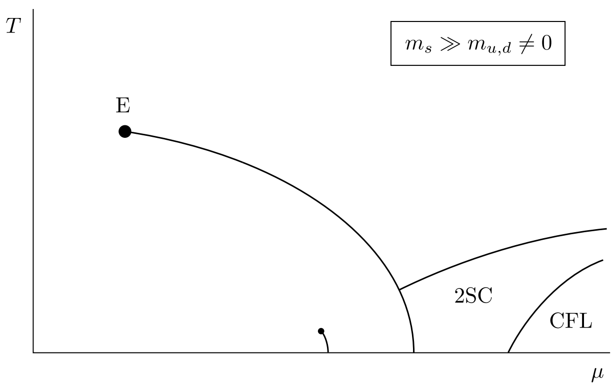

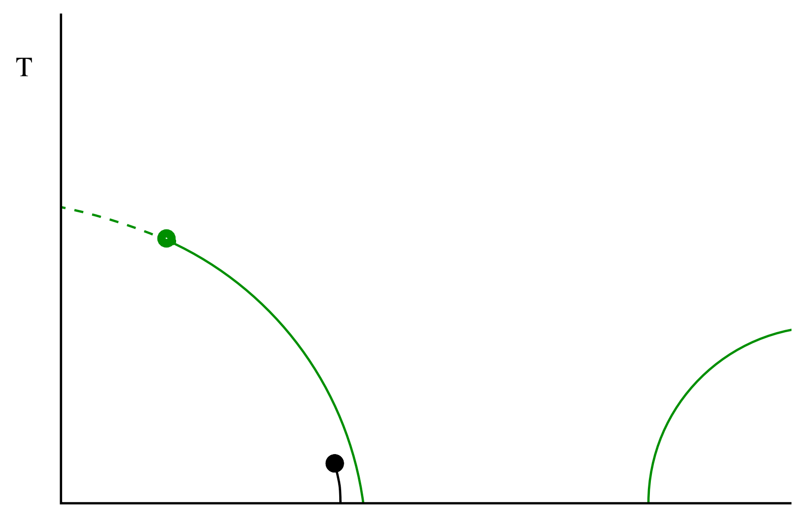

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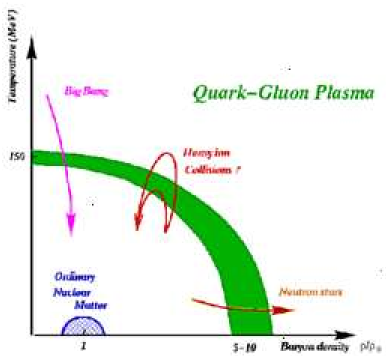

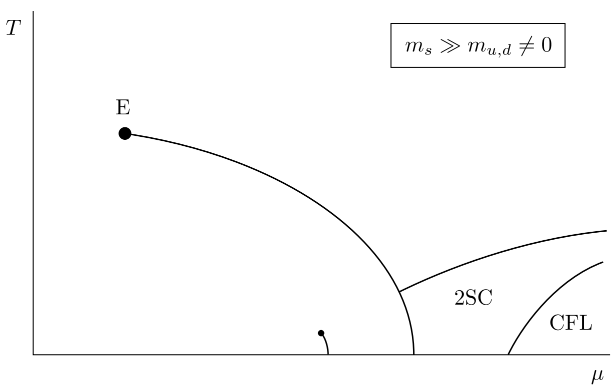

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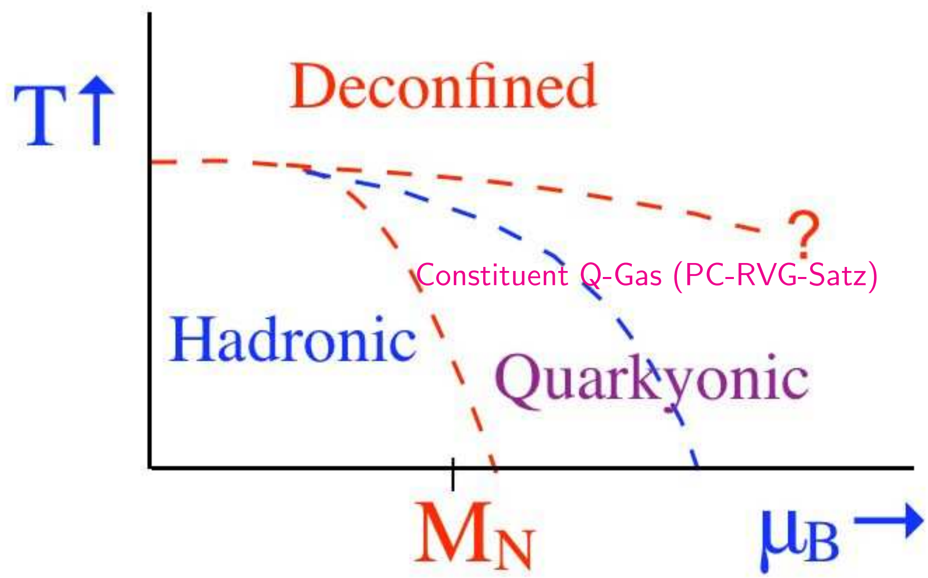
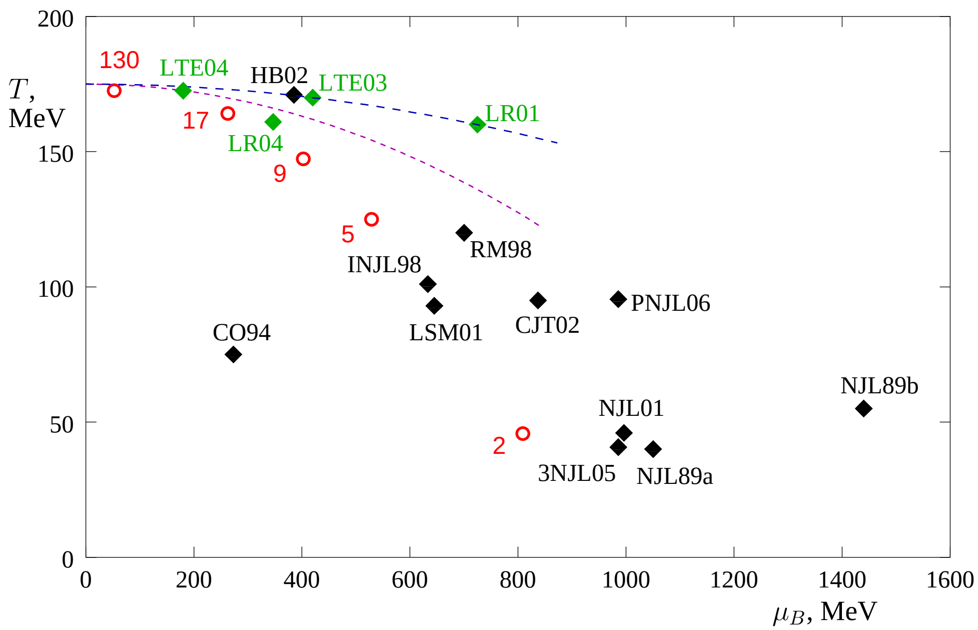


Models vs. Lattice QCD

- Many aspects of quark-gluon plasma (QGP) signals, including basic ideas underneath, depend on QGP properties, e.g., EoS, Debye Screening, etc.

- Computing in different models leads to different predictions.



M. Stephanov, Proceedings of Lattice 2007

- Large latent heat in Bag model, and hence long-lived mixed phase, is purely by construction, not at all generic.

- Lattice QCD, on the other hand, uses only the well-established QCD Lagrangian and the associated QFT knowledge/techniques.

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- **Same** lattice techniques used to fix the above, and to make predictions, such at T_c or EoS or Heavy Quark Diffusion constant.

- Use of path integrals at both zero temperature (for hadron masses) & finite temperature/density — key to lattice approach.

- $Z(T,\mu) = \mathrm{Tr}\ \exp -(\hat{H} - \mu\hat{N})/T \rightsquigarrow$ EoS, T_c,.. all thermodynamics. E.g. the energy density is $-V^{-1}\ \partial \ln Z/\partial(1/T)$.
  ($h/2\pi = c = k = 1$ used.)

QCD Thermodynamics from First Principles

- Simply use $\hat{H}_{QCD}$ and $\hat{N}_{baryon}$ above to obtain QCD thermodynamics from basics.

- Evaluation of trace : too complicated, intractable and even gauge dependent. Gauss' law has to be imposed.

- Reformulate the problem as a functional integral over all the fields in the theory of the gauge invariant $\mathcal{L}_{QCD}$ in Euclidean time.

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- A sketch of the idea : Let $\mu_B = 0$ and $\hat{H}_{QCD}$ be replaced by that of a one dimensional QM case. So, $\hat{H} = \hat{p}^2/2m + V(\hat{x})$.

- Use $\hat{x} \mid x\rangle = x \mid x\rangle$ as the complete and orthonormal set of states to evaluate the trace : $Z = \int dx \, \langle x \mid \exp(-\hat{H}/T) \mid x\rangle$.

- Let us divide $1/T = n\epsilon$, with ϵ very small, so that $\langle x \mid \exp(-\hat{H}/T) \mid x\rangle = \langle x \mid \prod_i^n \exp[-(\epsilon \hat{H})] \mid x\rangle$.

- Inserting the completeness relation n - 1 times & identifying $x_0 = x_n$, $Z = \int \prod dx_i \; \langle x_{i-1} \mid \exp[-\epsilon V(\hat{x})/2] \exp[-\epsilon \hat{p}^2/2m] \exp[-\epsilon V(\hat{x})/2] \mid x_i\rangle$.

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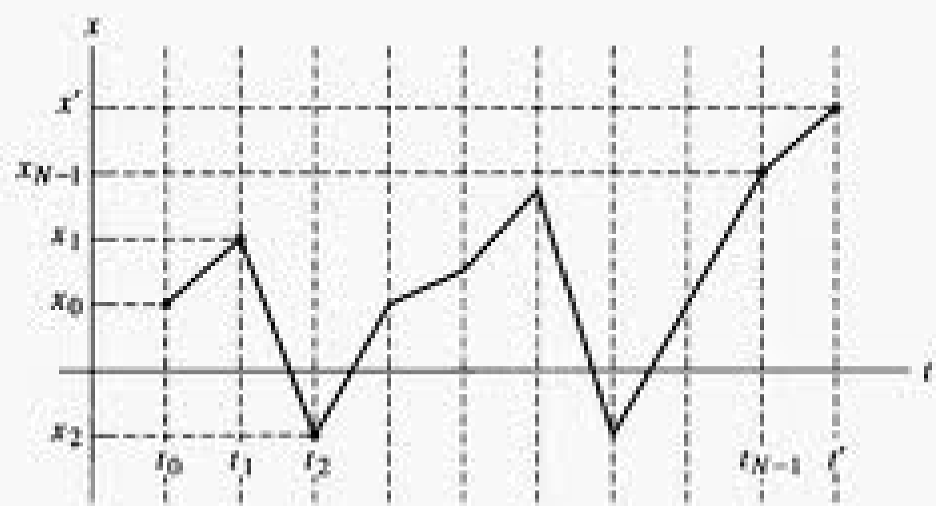

FIGURE 8.4
A possible path taken by the particle in going from position x_0 at time t_0 to a position x' at time t', with intermediate positions x_1 at time t_1, x_2 at time t_2, and so on.

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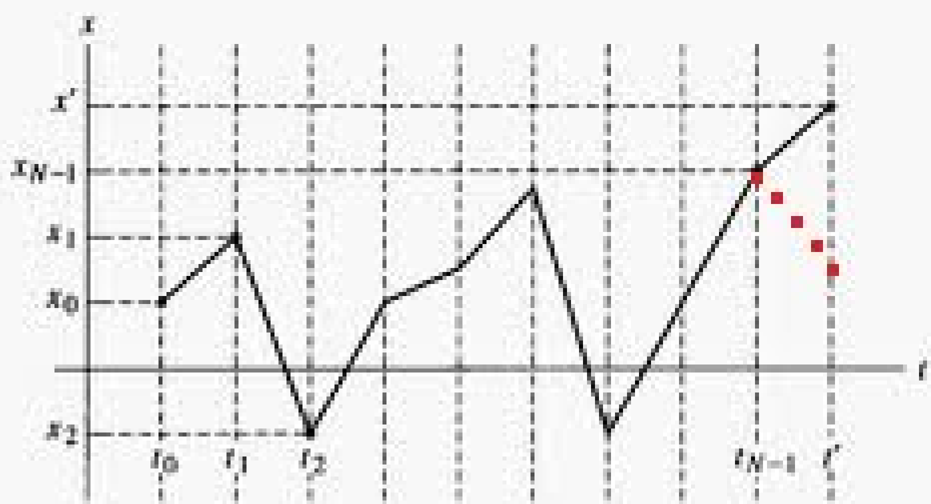

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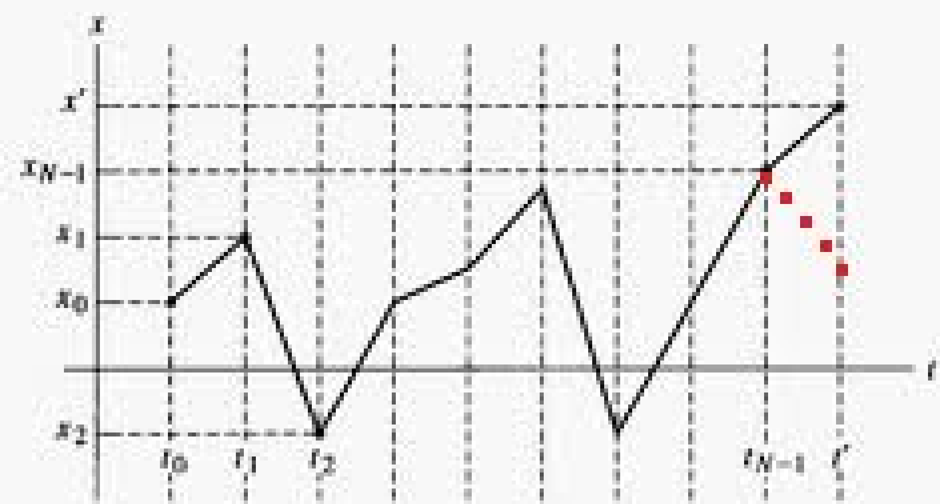

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- Noting that $\hat{P}$ is the canonical momentum operator has similar complete eigenstates too, these can be inserted twice in each term above.

- Evaluating the matrix elements, and simplifying, one obtains :
  $Z = \int \prod dx_i \ dp_i \ \exp[-\epsilon \Big[p_i^2/2m + V(x_i)/2 + V(x_{i-1})/2 - ip_i \frac{x_i - x_{i-1}}{\epsilon} \Big]$.

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- Integrating over p_i by completing the square,
  $Z = Const. \int_{x_0 = x_n} \prod dx_i \exp\left[- \frac{m}{2\epsilon}(x_i - x_{i-1})^2 - \epsilon \frac{V(x_i) + V(x_i)}{2} \right]$.

- Letting $\epsilon \to 0$, $Z = \sum_{paths}^{x(0)=x(1/T)} \exp(-S)$ where $S = \int_0^{1/T} d\tau [\frac{m}{2}\dot{x}^2 + V(x)]$.

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Remarks :

1. Z is now sum over all possible paths of e^{-S} : Euclidean path integral over the Euclidean time $\tau (= it)$.

2. In the limit $T \to 0$, the partition function Z reduces to a generating functional, i.e., the usual $T = 0$ path integral.

Generalization to Field Theory

- Recall that we introduced 'time' in form of $T^{-1} = n\epsilon$, introduced a transfer matrix from ith to $i+1$th time slice, and wrote $Z = \text{Tr } T^n$.

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- Following the same procedure for Scalar fields, with (ϕ, π) as conjugate pair, Quark fields with $(\psi, \bar{\psi})$, and/or Gauge fields with $(\vec{A}, \vec{E})$, one can obtain the Euclidean path integral for respective theories.

- $Z_{NS} = \int_{\phi(0)=\phi(1/T)} \mathcal{D}\phi \ \exp\left[- \int_0^{1/T} d\tau \int d^3x (\frac{\partial\phi}{\partial\tau})^2 + (\nabla\phi)^2 + m^2\phi^2 + V(\phi) \right]$
  $Z_{QCD} =$
  $\int_{\psi(0)=-\psi(1/T), \bar{\psi}(0)=-\bar{\psi}(1/T), A_\mu(0)=A_\mu(1/T)} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu \ \ e^{-\int_0^{1/T} \ d\tau d^3x \ \mathcal{L}_{QC\mathcal{D}}}$

Remarks :

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3. Fermion path integral defined in terms of Grassmann variables, satisfying $\eta_i\eta_j + \eta_j\eta_i = 0$. Note the anti-periodic boundary condition for (anti)quarks.

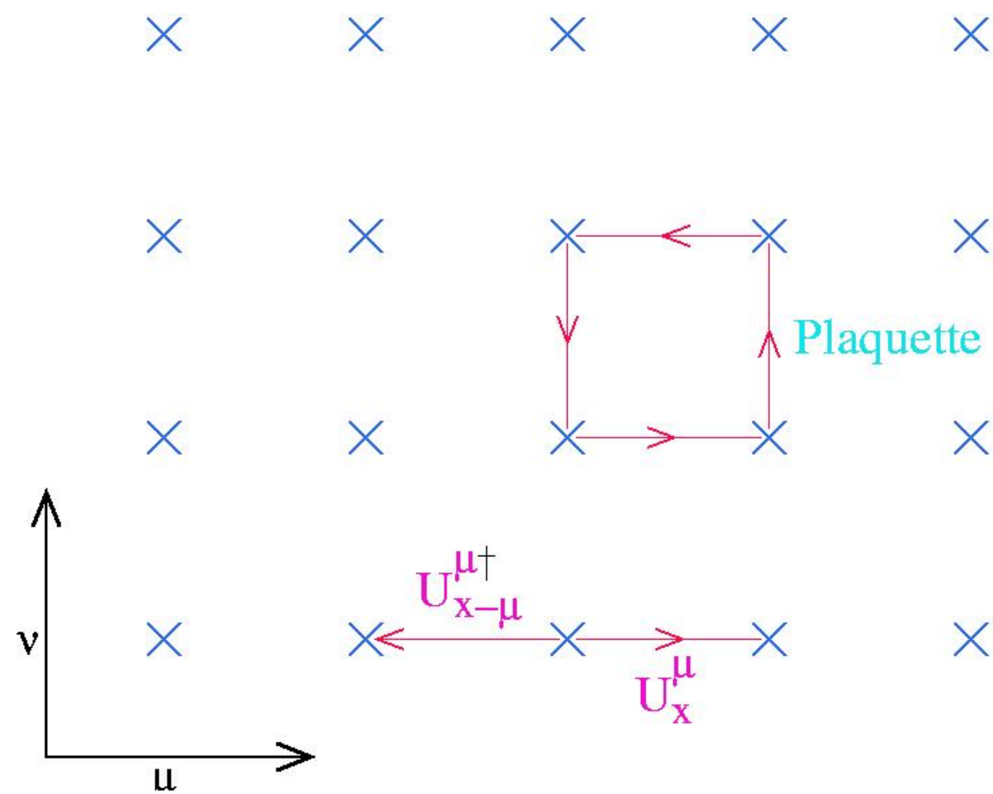
4. Scalar or vector fields (bosons) have periodic boundary conditions.

5. Manifest gauge invariance for Z_{QCD}.

6. Only Gaussian (i.e, free) field integrals are doable analytically.

Basic Lattice QCD

- Discrete space-time : Lattice spacing a UV Cut-off.
- Quark fields $\psi(x)$, $\bar{\psi}(x)$ on lattice sites.
- Gluon Fields on links : $U_\mu(x)$



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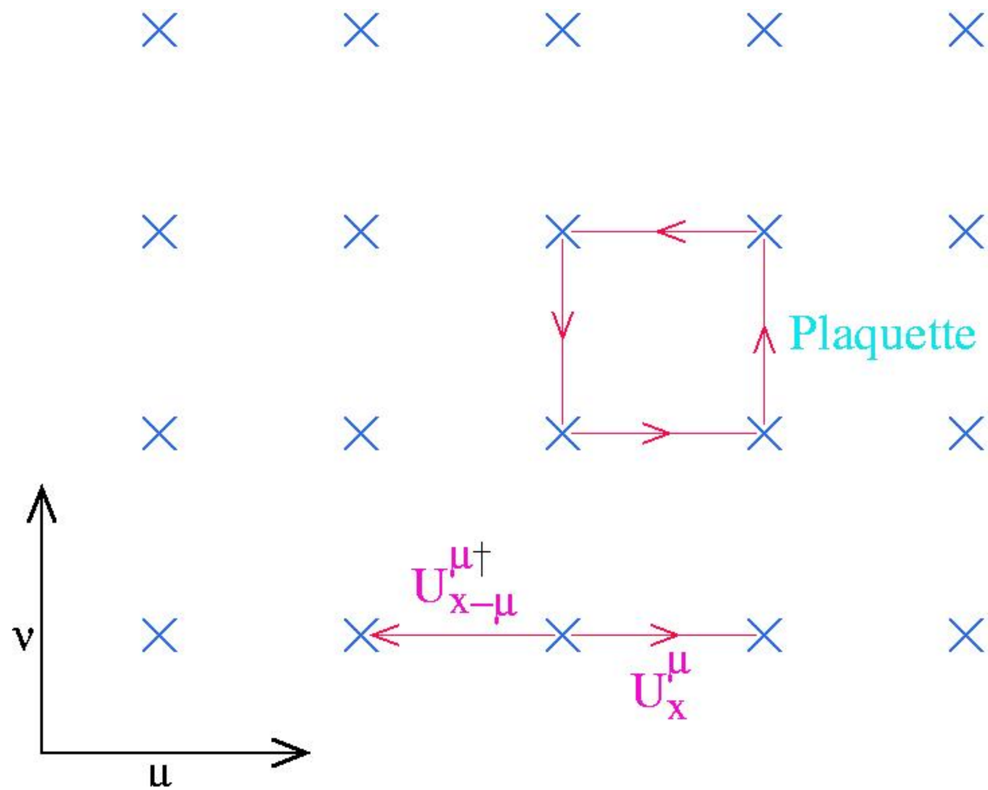
- Gauge invariance : Actions from Closed Wilson loops, e.g., plaquette.
- Fermion Actions : Staggered, Wilson, Overlap, Domain Wall..



Scalar fields

- Klein-Gordon equation (Euclidean time) : $(-\Box + m^2)\phi = 0$, where $\Box = \sum_\mu \partial_\mu \partial_\mu$ and $\mu = 1, 2, 3, 4$.

- Derived from the action, $S_E = \frac{1}{2} \int d^4x \; [(\partial_\mu \phi)^2 + m^2 \phi^2] = \frac{1}{2} \int d^4x \; [\phi(-\Box + m^2)\phi]$.

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- Add nontrivial $V(\phi) = \lambda\phi^4$: now only small λ can be tackled analytically, i.e., perturbation theory.

- Recall integrals are limits of sums : $I = \int f(x)dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x$.

- To perform the functional integral in general, indeed even to define it, employ a space-time grid : lattice field theory.

- Let $(\vec{x}, x_4) \to na \equiv (n_1 a_s, n_2 a_s, n_3 a_s, n_4 a_t)$, with $a_s(a_t)$ as lattice spacing in a space(time) direction. We will mostly use $a_s = a_t = a$.

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- Volume is $V = N_s^3 a_s^3$ & temperature is $T = (N_t a_t)^{-1}$.

- Define forward and backward differences to replace derivatives. $\Delta_\mu^f \phi(x) = \frac{(\phi(x+a\hat{\mu})-\phi(x))}{a}$ and $\Delta_\mu^b \phi(x) = \frac{(\phi(x)-\phi(x-a\hat{\mu}))}{a}$.

- $Problem$: Show that i) $(\Delta_\mu^f)^\dagger = -\Delta_\mu^b$ and ii) $\Box = \sum \Delta_\mu^b \Delta_\mu^f$ with $\Box\phi(x) = a^{-2} \sum_\mu \{ \ \phi(x+\hat{\mu}a) + \phi(x-\mu a) - 2\phi(x)\} \equiv a^{-2}\hat{\Box}\phi(x)$.

- Note that ϕ has mass dimension one. Define $\hat{\phi}_n = a\phi(x) \equiv a\phi(na)$.

- Continuum action can be now written in terms of lattice variables.
$S_E = \frac{1}{2}\int d^4x\ \phi(x)(-\Box + m^2)\phi(x) \longrightarrow$
$S_{lat} = \frac{1}{2}a^4\sum_n \frac{\hat{\phi}_n}{a}(-\frac{\hat{\Box}}{a^2} + m^2)\frac{\hat{\phi}_n}{a} \quad = \frac{1}{2}\sum_n \hat{\phi}_n(-\hat{\Box} + \hat{m}^2)\hat{\phi}_n$

- We thus obtain the lattice action in terms of dimensionless variables, parameter and 'operators'. Here $\hat{m} = a\ m$ is dimensionless mass.

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- We thus obtain the lattice action in terms of dimensionless variables, parameter and 'operators'. Here $\hat{m} = a\ m$ is dimensionless mass.

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Comments :

1. Generically true for all lattice field/gauge theories, including Lattice QCD, that action is defined by dimensionless field variables, and parameters.

2. Lorentz symmetry, which became the $O(4)$ symmetry in the Euclidean space, is badly broken on the lattice. Only discrete rotations by $\pi/2$ are symmetries of the action. Lattice also breaks translational invariance.

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4. Above discretization is by no means unique but only simple. Indeed, Any discretization could be chosen as long as $\lim_{a\to 0} S_E^{lat} = S_E$ in the continuum limit.

5. Infinitely many lattice actions possible which all reproduce the continuum action as $a \to 0$. Physics demands that they all lead to the same result for QFT : **Universality** and *Improved actions*.

6. Interactions like $\lambda\phi^4$ can be added in a straightforward way : $\lambda\hat{\phi}(n)^4$. In general, any $V(\phi)$ is added this way.

Lattice Propagator

- Introduce (dimensionless) lattice sources, and compute the propagator as usual by taking derivatives with respect to them.

- Let $M_{n,m} = -\sum_{\mu}[\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}] + \hat{m}^2\delta_{n,m}$, then $Z[J] = \int \prod_n d\hat{\phi}_n \;\; e^{-\frac{1}{2}\Phi^\dagger M \;\; \Phi + \sum_n \hat{J}_n \hat{\phi}_n}$, where Φ is a column vector of $\hat{\phi}_n$.

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- Complete square and integrate over each $\hat{\phi}_n$ to obtain the lattice propagator $\hat{G}(n, m; \hat{m})$.
$Z[J] = \frac{e^{\frac{1}{2}\sum_{n,m} \hat{J}_m {M^{-1}}_{mn} \hat{J}_n}}{\sqrt{\det \;\; M}} \Longrightarrow \hat{G}(n, m; \hat{m}) \equiv \langle \hat{\phi}_n \hat{\phi}_m \rangle = {M^{-1}}_{nm}$.

- *Problem :* Show that in the Fourier space $M(\hat{p}_\mu) = 4\sum_\mu \sin^2(\hat{p}_\mu/2) \;+\; \hat{m}^2$, with discrete momenta $\hat{p}_\mu = 2\pi n_\mu / N_\mu$ for $n_\mu =$ 0,1..., $N_\mu - 1$ & $N_\mu = N_s(N_t)$.

- Correlation function $\langle \hat{\phi}_n \hat{\phi}_m \rangle \sim f(\mid n - m \mid)$, typically exponential/power law decay. Correlation length $\hat{\xi} \sim \hat{m}$ pole of the propagator.

- Since typically $\hat{X} = aX$, continuum limit $a \to 0$ of $\hat{G}$ is non-trivial only if some dimensional quantity such as m is fixed in physical units and $\hat{m} \to 0$.

- Equivalently, lattice correlation length in $\hat{\xi} = \xi/a \to \infty \Longrightarrow$ Tune coupling(s) to second order phase transition for the lattice theory. How ?

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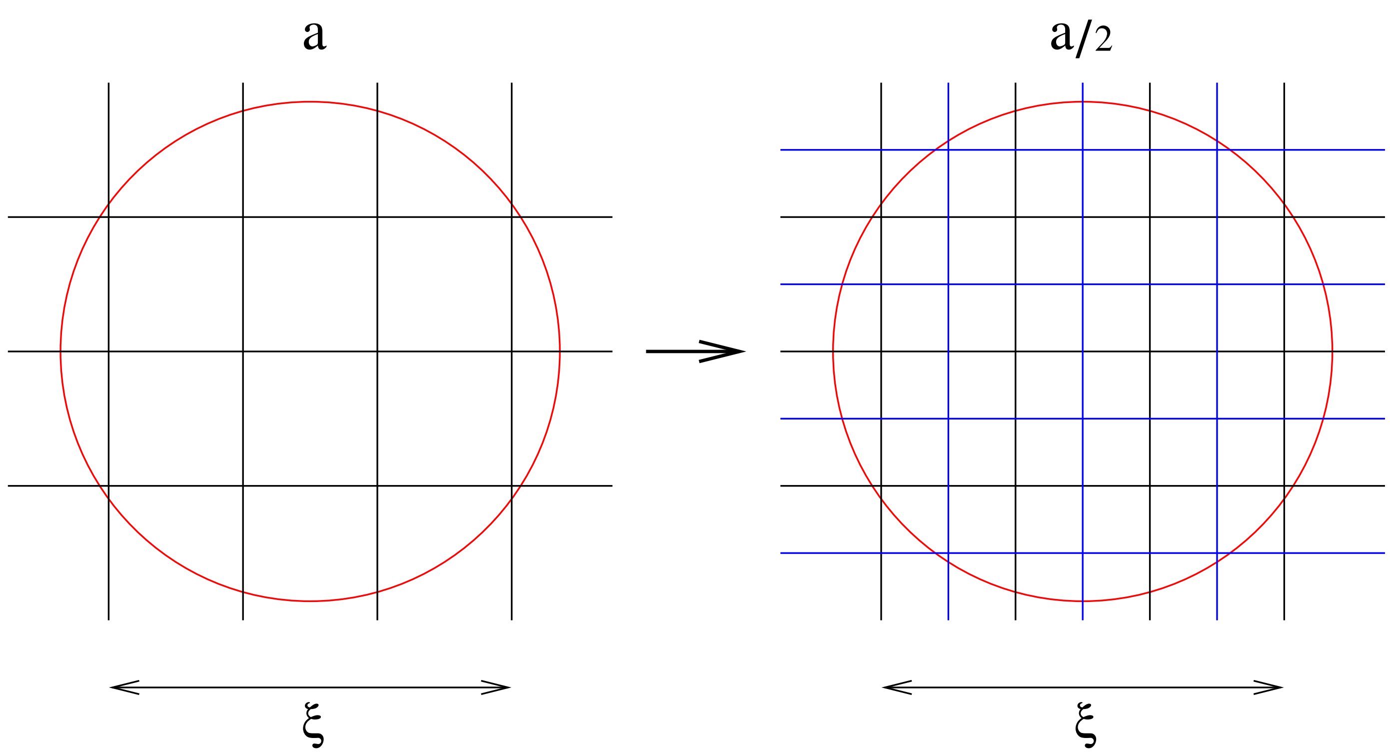

- We need to hold $x = na$ and $y = ma$ constant as $a \to 0 \Longrightarrow N_s, N_t \to \infty$.

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  $G(x, y; m) = \lim_{a\to 0}\ a^2\ \int_{-\pi/a}^{\pi/a} d^4p \frac{e^{ip\cdot x}}{\sum_\mu 4\sin^2\frac{p_\mu a}{2} + m^2a^2}$

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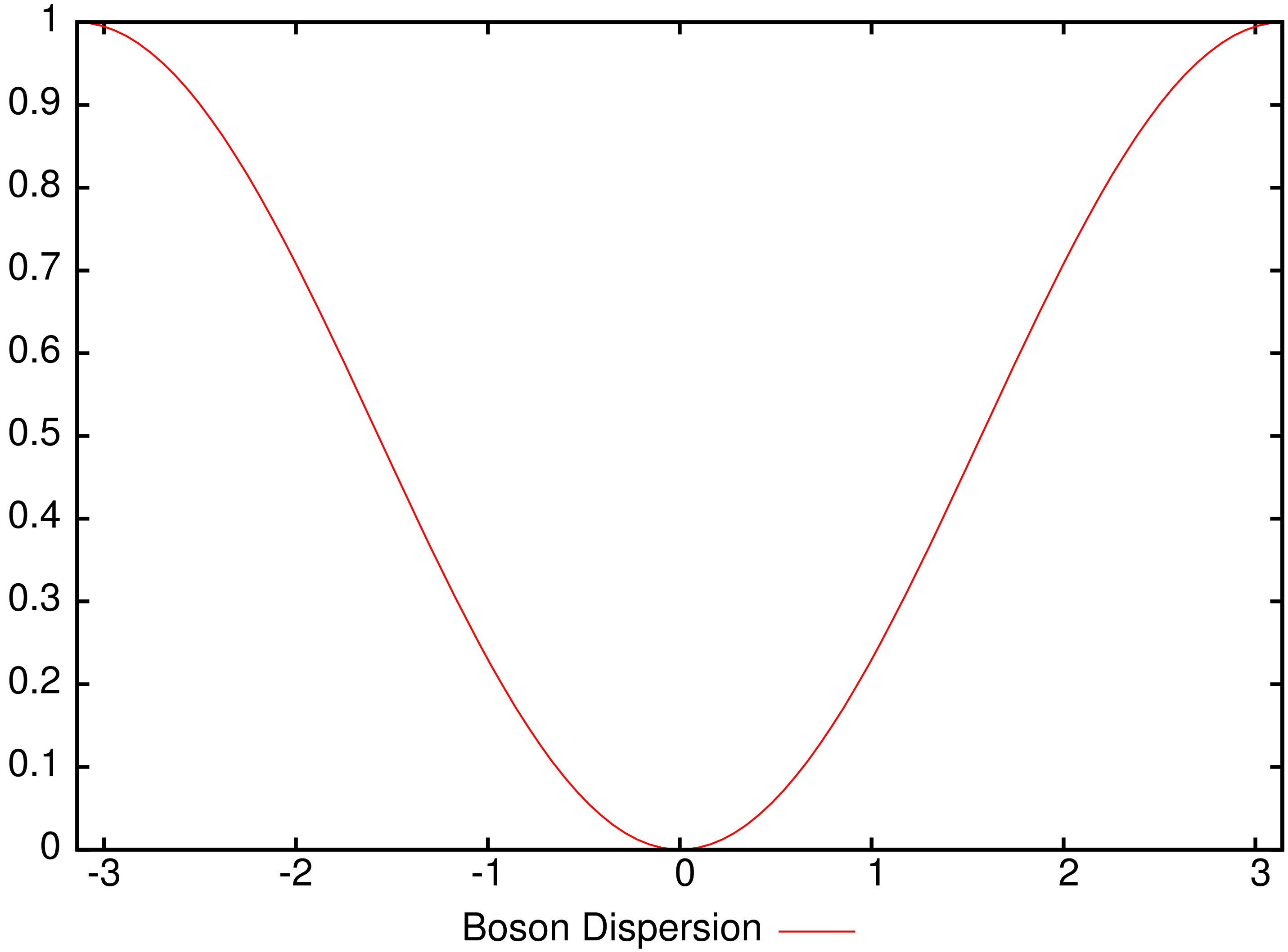
- Thus holding physical mass m constant, leads to the correct 'quantum continuum' limit in the sense that the correct correlator/propagator results.



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-3
-2
-1
0
1
2
3
Boson Dispersion

Fermions on Lattice

Bringing in Interactions

Continuum Limit