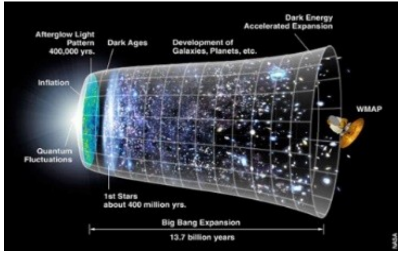

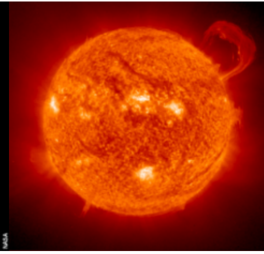
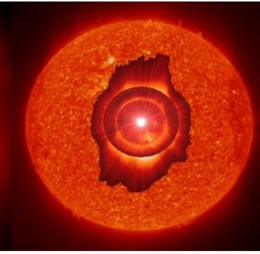
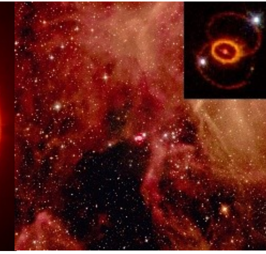
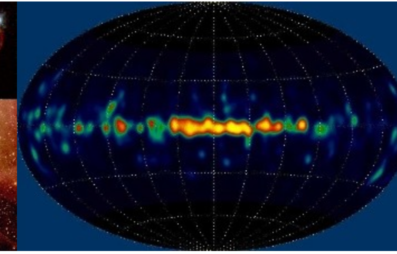
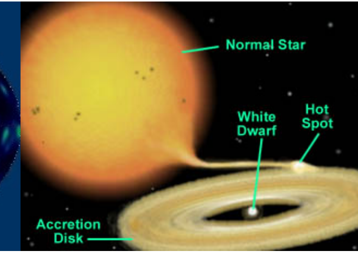

Stellar Reaction Rates

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Plan of lecture I

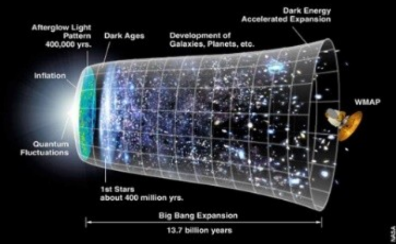
I. Introduction : Nuclear astrophysics & Nuclei in the Cosmos

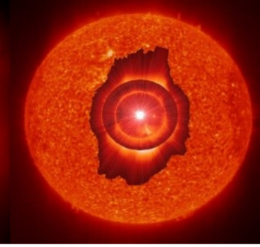
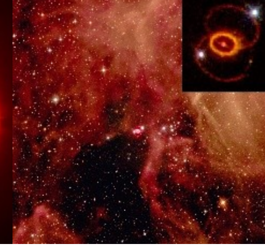
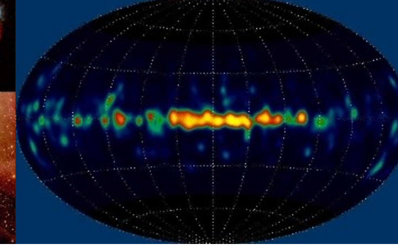
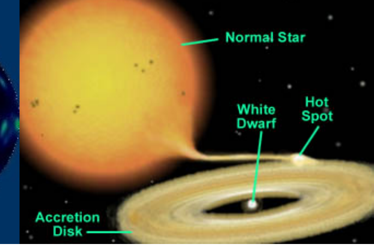
II. Nuclear reaction cross sections:
   - Definitions
   - Quantum tunneling,
   - Astrophysical S-factor
   - Reaction mechanisms (non-resonant & resonant processes)

III. Thermonuclear reaction rates
   - Definitions

Text books

J. Audouze and S. Vauclair	An introduction to Nuclear Astrophysics D. Reidel Publishing Company, Dordrecth, 1980
D.D. Clayton	Principles of stellar evolution and nucleosynthesis The University of Chicago Press, 1983
C.E. Rolfs and W.S. Rodney	Cauldrons in the Cosmos Thee University of Chicago Press, 1988 (…the “Bible”)
C. Illiadis	Nuclear Physics of Stars Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, 2007



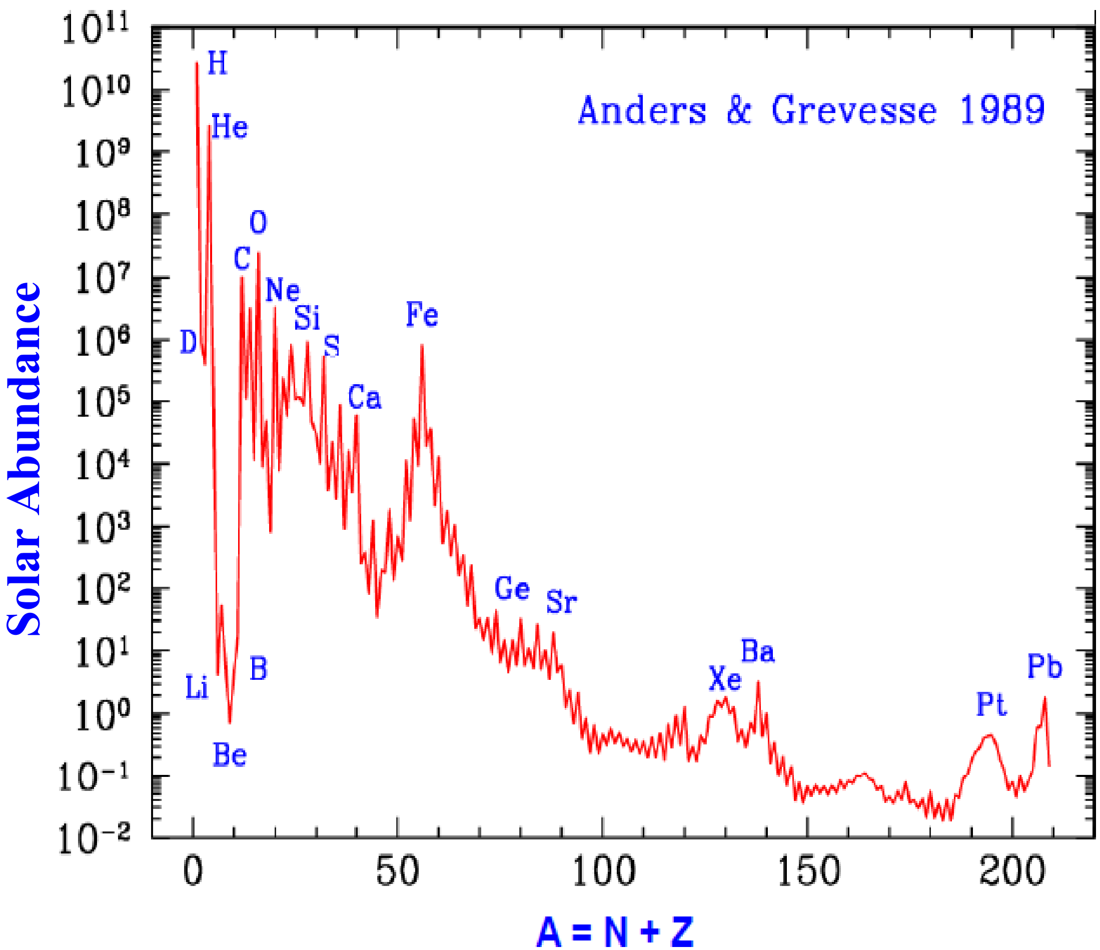






NUCLEAR ASTROPHYSICS

Nuclear astrophysics is the science which addresses some of the most compelling questions in nature:

- How do stars form and evolve?
- What powers the stars?
- What is the origin of the chemical elements present in our Universe?
- Which nucleosynthesis processes are responsible of the observed solar abundances?

Abundance curve of the elements:



Data sources:

Earth, Moon, meteorites, solar & stellar spectra, cosmic rays...

Characteristics:

- 12 orders-of-magnitude span
- H ~ 75%
- He ~ 23%
- C → U ~ 2% ("metals")
- D, Li, Be, B under-abundant
- O the 3rd most abundant
- C the 4th most abundant
- exponential decrease up to Fe
- peak near Fe
- nearly flat distribution beyond Fe with some peaks

Introduction: Nuclei in the Cosmos

The answer to all the questions concerning the stars and the origin of the nuclei in the cosmos is given by the interaction of three fields:

Observations (astronomy & geology)

Nuclear Physics
Synthesis of nuclei

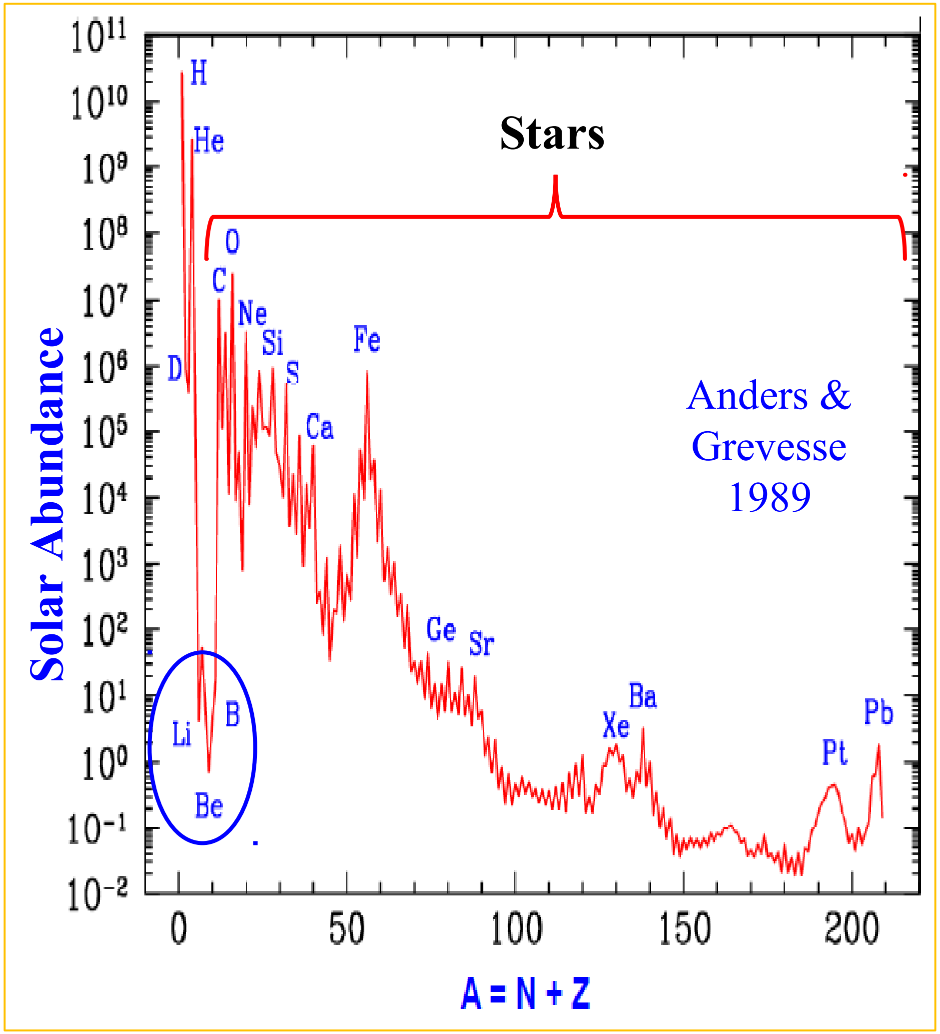
Astrophysics modelling (Big-Bang & stellar)

Nucleosynthesis: When and where?

➢**H, D, He, ^{7}Li$^{\#}$**
→ primordial nucleosynthesis
(Big-Bang) (A. Coc lecture)

➢**Li$^{\#}$, Be, B**
→ Cosmic ray spallation in Inter-Stellar Medium (ISM) : heavier and abondant nuclei (CNO) broken by interaction with p or α particle (A. Coc lecture)

➢**C, N, O ..., Fe, ... Pb,…**
→in star (calm & explosive)
(A. Coc lecture)



From nuclear physics to abundances

Improving the knowledge of the nucleosynthesis processes at work in the universe & the understanding of stellar evolution

Nuclear physics Experiments & theory
(cross-sections, resonance parameters, masses, β-decays,…)

Reaction rate

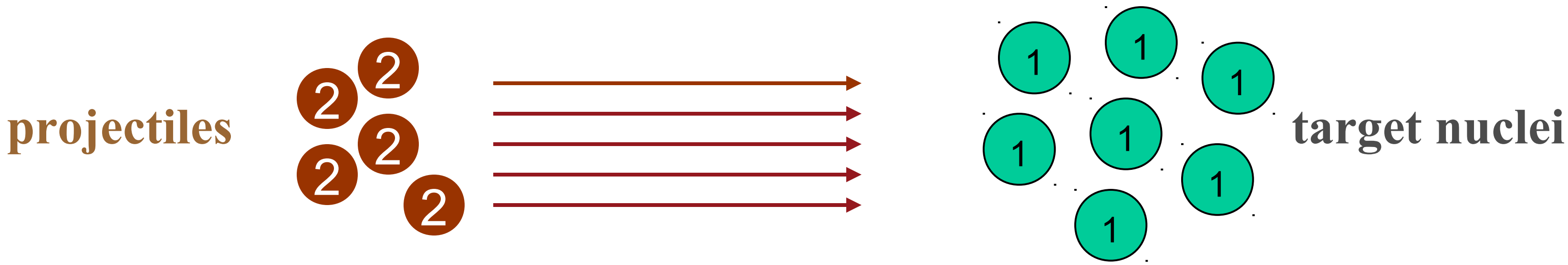
Astrophysics Modelling
Network calculations
(BBN & stellar evolution modelling, nucleosynthesis)

Abundances

Observations
(On earth, meteorites, satellites,…)

Abundances

Nuclear reaction cross sections: Definitions



- Cross section of the reaction 1 + 2 → 3 + 4 [notation 1(2,3)4]:

$$\sigma(\text{cm}^2) = \frac{\text{Number of reactions/second}}{(\text{Number of incident particles/cm}^2\text{/second})(\text{number of target nuclei within the beam})}$$

= surface presented by 1 to the projectile 2 for a given reaction

- <u>"Billiard-type" description</u> of the cross section:

$\sigma = \pi(R_1 + R_2)^2$ with the nuclear radius $R_N \approx 1.3A^{1/3}$ fm (10^{-13} cm)

⇒ $\sigma(^1\text{H}+^1\text{H}) = 0.2\times10^{-24}$ cm^2, $\sigma(^{238}\text{U}+^{238}\text{U}) = 4.8\times10^{-24}$ cm^2

⇒ unit of nuclear cross sections: 1 barn (b) = 10^{-24} cm^2

Nuclear reaction cross sections: The maximum cross-section

- Quantum description of the maximum reaction cross section:

$$\sigma_{\max} = (2l+1)\pi\lambda\!\!\!^{-2} \qquad \text{where} \quad \lambda\!\!\!^{-} = \frac{\hbar}{\sqrt{2\mu E}} = \frac{m_1 + m_2}{m_1}\frac{\hbar}{\sqrt{2m_2 E_2}}$$

is the de Broglie wavelength, E the total kinetic energy in the centre-of-mass system of reference, and $\mu = (m_1 m_2)/(m_1 + m_2)$ the reduced mass. Note that $\sigma_{max} \propto 1/E$

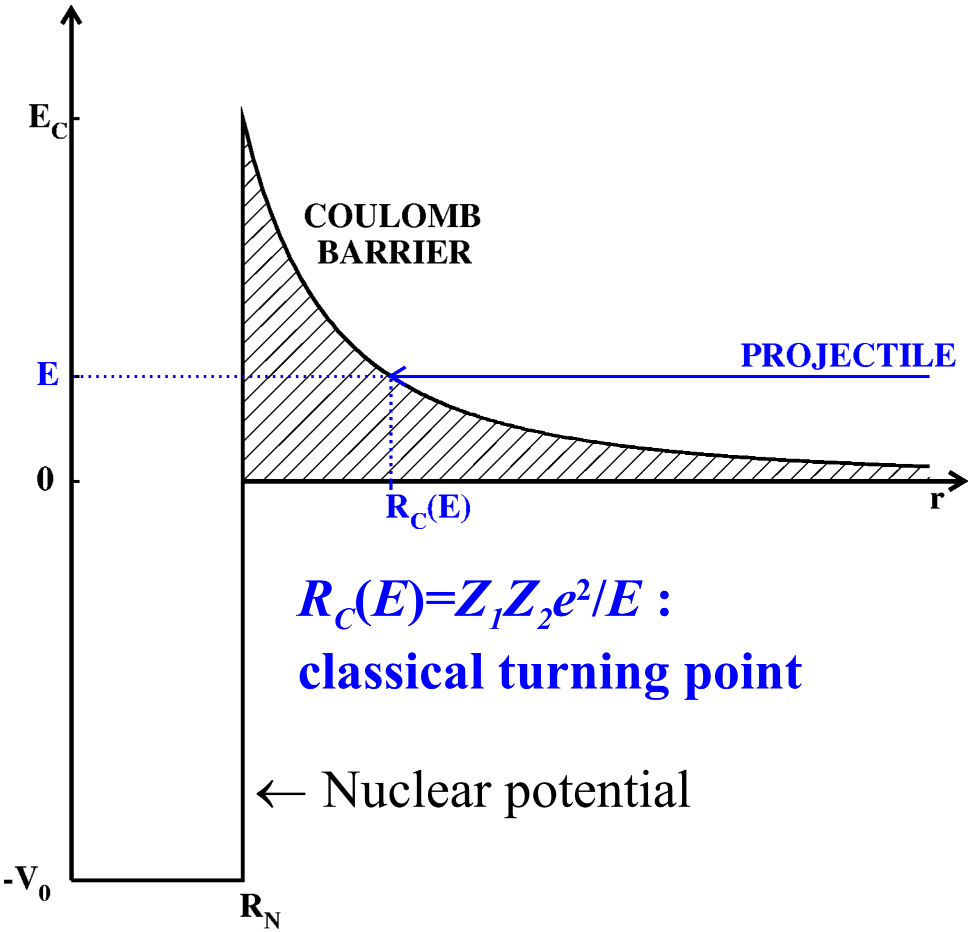
The statistical factor $(2l+1)$ corresponds to the number of eigenstates of the system 1 + 2 of angular momentum $\boldsymbol{L}$ (l is the orbital quantum number)

- $\sigma < \sigma_{max}$ in part. because of the centrifugal and Coulomb barriers

- Centrifugal barrier: energy needed to move closer 1 and 2 to a distance r given the orbital angular momentum L (classical mech.)

$$V_{\text{cent}}(r) = \frac{\left\|\vec{L}\right\|^2}{2\mu r^2} \Rightarrow V_{\text{cent}}(r) = \frac{l(l+1)\hbar^2}{2\mu r^2} \quad l(l+1)\hbar: \ \text{eigenvalues of } \boldsymbol{L}^2$$

Nuclear reaction cross sections: The Coulomb barrier



$R_C(E)=Z_1Z_2e^2/E$ **:**
classical turning point

← Nuclear potential

- In a reaction between charged nuclei (atomic numbers Z_1, Z_2)

$$\boxed{V_{\mathrm{coul}}(r) = \frac{Z_1Z_2e^2}{r} = \frac{1.44Z_1Z_2}{r\ (\text{in fm})}\ \text{MeV}}$$

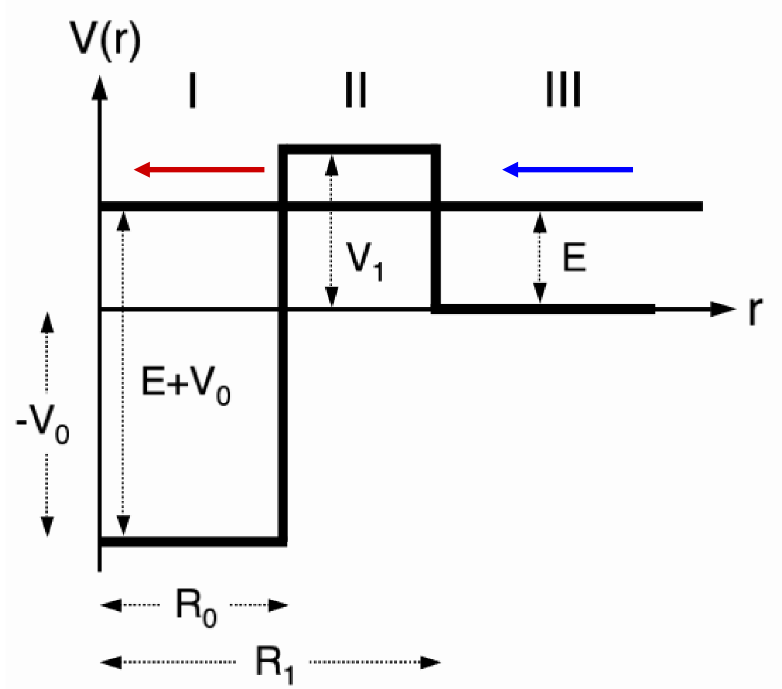
- In stars, $T_C \sim 10^7$ - 10^9 K

$\Rightarrow kT_{\mathrm{C}} \sim$ 1-100 keV $< \mathrm{V}_{\mathrm{coul}}(R_N)$

$\Rightarrow$ penetration of the Coulomb barrier by the "tunnel effect" (quantum mechanic effect)

Nuclear reaction cross sections: The tunnel effect (1)

- <u>Square-barrier potential with $l = 0$</u> The radial wave functions $\phi(r)$ (1D) are solution of the time-independent Schrödinger equation:



$$\boxed{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+V\right]\phi(r)=E\phi(r)}$$

$$\Rightarrow \phi_{III}(r)=Fe^{ikr}+\boxed{Ge^{-ikr}}\text{ with } k^2=\frac{2\mu}{\hbar^2}E$$

$$\Rightarrow \phi_{II}(r)=Ce^{-\kappa r}+De^{\kappa r}\text{ with } \kappa^2=\frac{2\mu}{\hbar^2}\left(V_1-E\right)$$ $\Leftarrow$ vanishing waves

$$\Rightarrow \phi_{I}(r)=Ae^{iKr}+\boxed{Be^{-iKr}}\text{ with } K^2=\frac{2\mu}{\hbar^2}\left(E+V_0\right)$$ $\Leftarrow$ plane waves

Nuclear reaction cross sections: The tunnel effect (2)

- Wave function matching conditions at the boundaries:

$$(\phi_I)_{R_0} = (\phi_{II})_{R_0} \qquad (\phi_{II})_{R_1} = (\phi_{III})_{R_1}$$

$$\left(\frac{d\phi_I}{dx}\right)_{R_0} = \left(\frac{d\phi_{II}}{dx}\right)_{R_0} \qquad \left(\frac{d\phi_{II}}{dx}\right)_{R_1} = \left(\frac{d\phi_{III}}{dx}\right)_{R_1} \quad \Rightarrow \; B = f(G)$$

- Transmission coefficient of the barrier:

$$\boxed{T = \frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{K|B|^2}{k|G|^2}}$$

given the incident and transmitted current densities (or fluxes)

$$j_{\text{inc}} = v_{III}|G|^2 = \frac{\hbar k}{\mu}|G|^2 \quad \text{and} \quad j_{\text{trans}} = v_I|B|^2 = \frac{\hbar K}{\mu}|B|^2$$

We finally obtain: $\boxed{T \approx \exp\left[-(2/\hbar)\sqrt{2\mu(V_1 - E)}(R_1 - R_0)\right]}$

- Numerical application: p + p interaction at E = 100 keV

$R_0 = 1.3\times2 = 2.6$ fm, $R_1 \equiv R_C = 14.4$ fm, $V_1 \equiv \mathrm{V}_{\text{coul}} = 550$ keV $\Rightarrow T = 9\%$

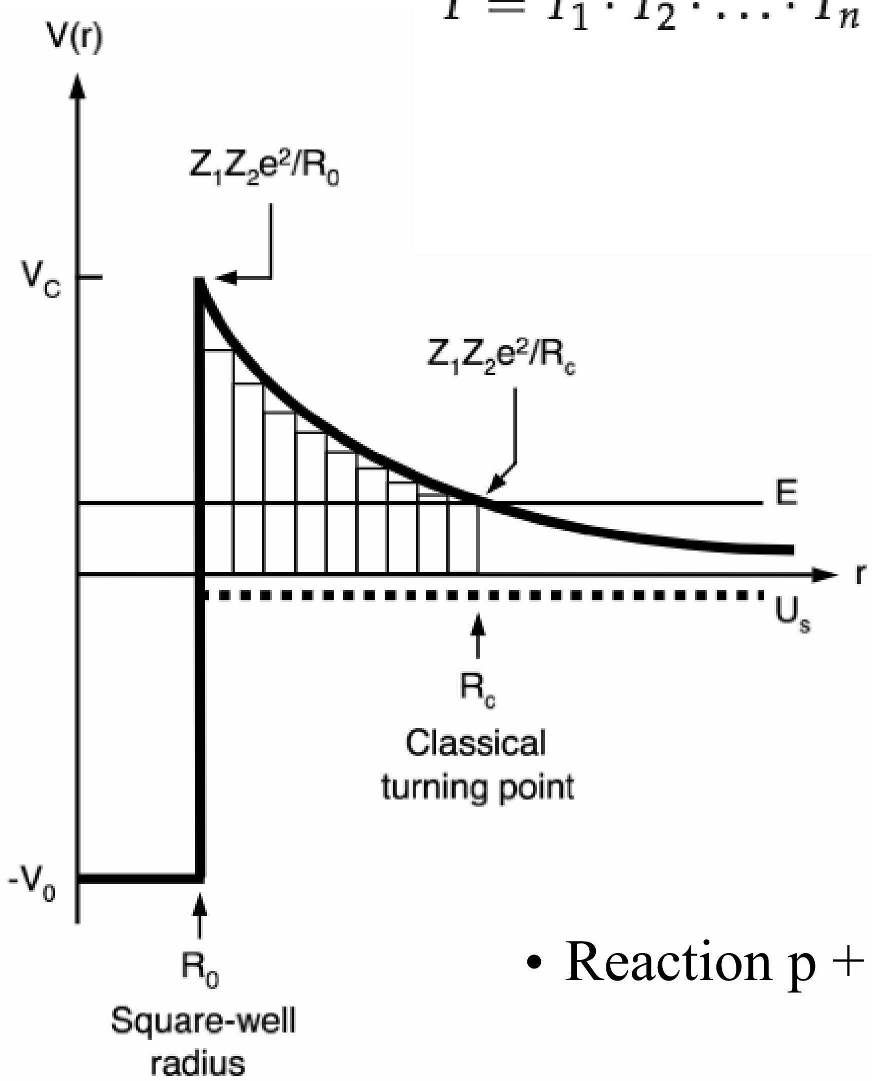
With $V_1 = 36.8$ MeV corresponding to V_{cent} for $l = 2 \Rightarrow T = 2\times10^{-10}$

Nuclear reaction cross sections: The tunnel effect (3)

- <u>Transmission coefficient of the Coulomb barrier</u>

$$\hat{T} = \hat{T}_1 \cdot \hat{T}_2 \cdot \ldots \cdot \hat{T}_n \approx \exp\left[-\frac{2}{\hbar}\sum_i \sqrt{2m(V_i - E)}(R_{i+1} - R_i)\right]$$

$$\xrightarrow[n \text{ large}]{} \exp\left[-\frac{2}{\hbar}\int_{R_0}^{R_c} \sqrt{2m[V(r) - E]}\, dr\right]$$



$$\boxed{T \approx \exp\left[-\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{2E}}Z_1Z_2e^2\right] = \exp(-2\pi\eta)}$$

η: Sommerfeld parameter

$\exp(-2\pi\eta)$: Gamow factor

$$2\pi\eta = 31.29 Z_1 Z_2 \sqrt{\frac{\mu_{\text{amu}}}{E_{\text{keV}}}}$$

- Reaction p + p (μ_{amu}=1/2) at $E_{keV} = 100 \Rightarrow$ <u>$T = 11\%$</u>

at $E_{keV} = 6 \Rightarrow$ <u>$T = 0.01\%$</u>

Nuclear reaction cross sections: The astrophysical S-factor

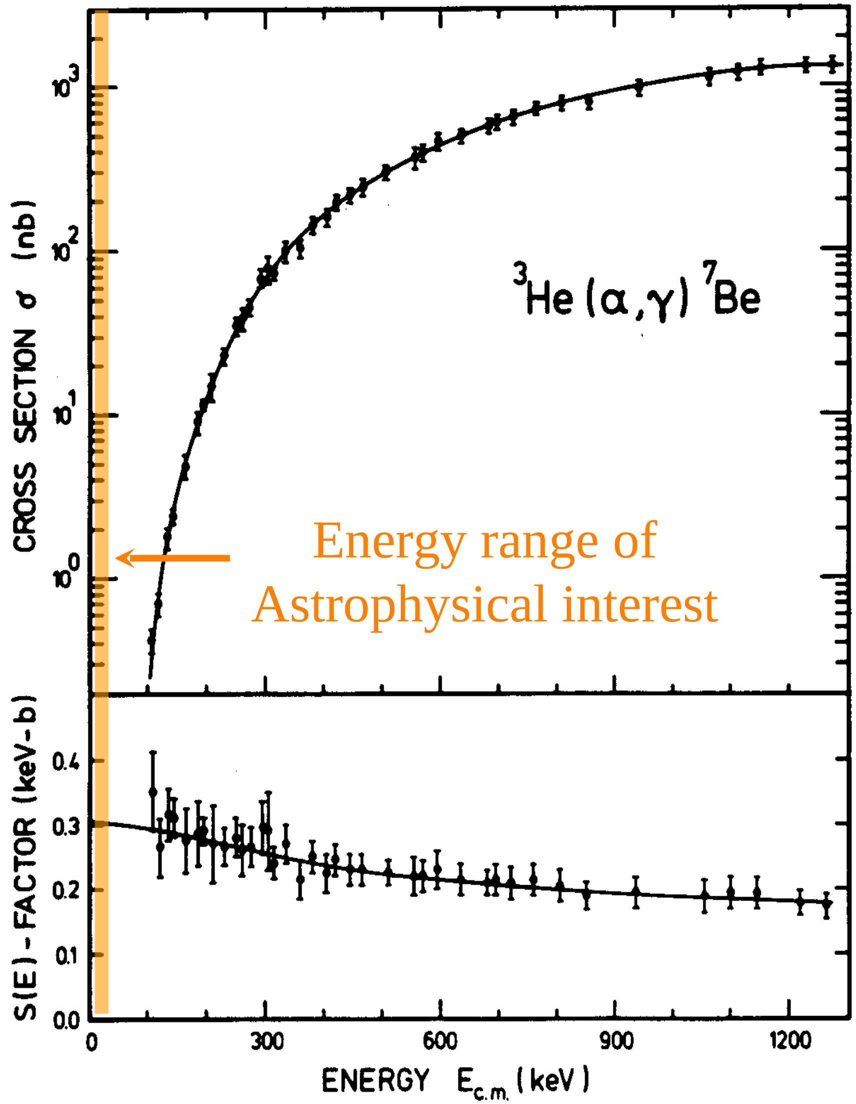
$$\sigma(E) = \frac{1}{E} \times e^{-2\pi\eta} \times S(E)$$

correction of the effect $\sigma_{\max}(E) \propto \lambda^2$

correction of the tunneling probability (l=0)

$S(E)$: astrophysical S-factor which contains all the nuclear effects for a given reaction

➢ (Sometimes) a smoothly varying function $\Rightarrow$ extrapolation to astrophysical energies



Nuclear reaction cross-sections: Reaction mechanisms

Consider reaction: $a + X \rightarrow b + Y$ (b = particle or photon)

Non-resonant process

One-step process leading to final nucleus Y $\sigma \propto | < b{+}Y \,|\, H \,|\, a{+}X > |^{2}$

single matrix element

- occurs at all interaction energies
- cross section has relatively WEAK energy dependence

Resonant process

Two-step process: 1) compound nucleus formation $a + X \rightarrow C^{*}$

2) decay of compound nucleus $C^{*} \rightarrow b + Y$

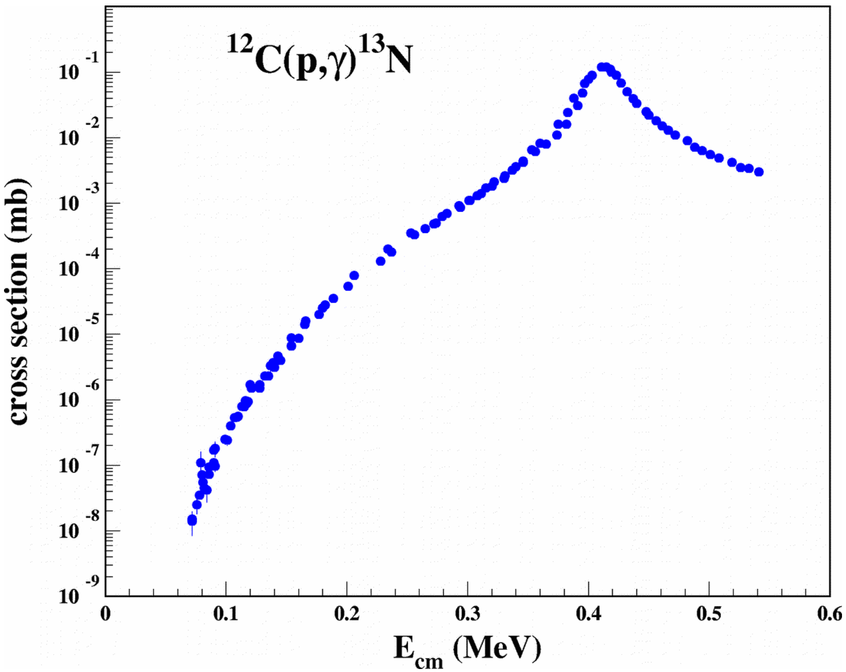
$$\sigma \propto | < b{+}Y \,|\, H' \,|\, C^{*} > |^{2} \; | < C^{*} \,|\, H \,|\, a{+}X > |^{2}$$

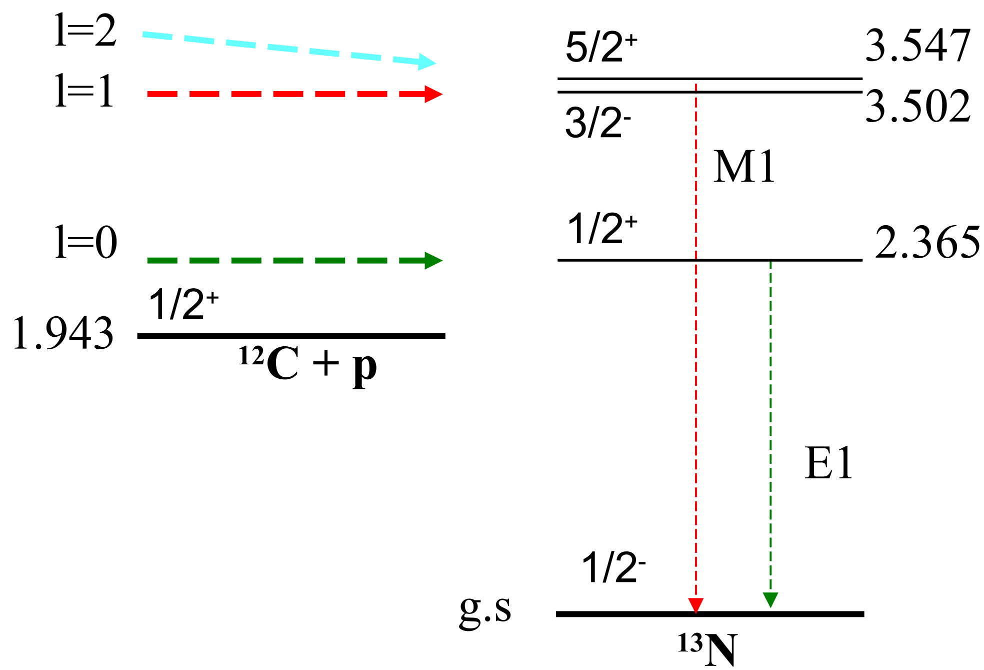
two matrix elements

- occurs at specific energies
- cross section has STRONG energy dependence

Nuclear reaction cross sections: Nuclear resonances

- **<u>A simple case</u>: $^{12}C(p,\gamma)^{13}N$**





- <u>Reaction Q-value</u> (Δ=mass excess):
- $Q = \Delta(^{12}C) + \Delta(p) - \Delta(^{13}N) = 1.943$ MeV
- $E_R = 2365 - 1943 =$ <u>422 keV</u>

- $\boldsymbol{J}_R = \boldsymbol{J}(^{12}C) + \boldsymbol{J}(p) + \boldsymbol{L} = 1/2, (-1)^l = 1$
- $\Rightarrow$ <u>$l = 0$</u>

Nuclear reaction cross sections: Nuclear resonance profile

- Energy profile of excited nuclear states:

  - Time-dependent wave function: $$\psi(t) = \psi(0)\exp\left(-\frac{i}{\hbar}E_R t\right)\times\exp\left(-\frac{t}{2\tau}\right)$$

  where τ is the mean lifetime of the excited state

  - The wave function as a function of energy is obtained by the Fourier transform (conjugate variables): $$\varphi(E) = \int_0^\infty \psi(t)\exp\left(\frac{i}{\hbar}Et\right)dt$$

  - The probability distribution is then:

$$f_R(E) = \left|\varphi(E)\right|^2 = \frac{\hbar}{2\pi\tau}\frac{1}{(E-E_R)^2 + (\hbar/2\tau)^2}$$

= Breit-Wigner profile (Cauchy-Lorentz distribution)

- Full width at half maximum: $$\Gamma = \frac{\hbar}{\tau}$$ $\Leftarrow$ Heisenberg uncertainty principle

Nuclear reaction cross sections: Partial width

- Partial width (energy units): $\Gamma_a = \hbar\lambda_a$ where λ_a is the probability per second that the decay particle a (≡ p, n, α, β…) crosses an imaginary sphere at the distance $r \to \infty$:

$$\lambda_a = \lim_{r\to\infty} v \iint_{\theta,\phi} |\psi(\mathrm{r},\theta,\Phi)|^2 r^2 \sin\theta d\theta d\Phi$$

$$\lambda_a = \lim_{r\to\infty} v \iint_{\theta,\phi} \left|\frac{\phi_l(r)}{r}\right|^2 \left|Y_l^m(\theta,\phi)\right|^2 r^2 \sin\theta d\theta d\phi = v|\phi_l(\infty)|^2$$

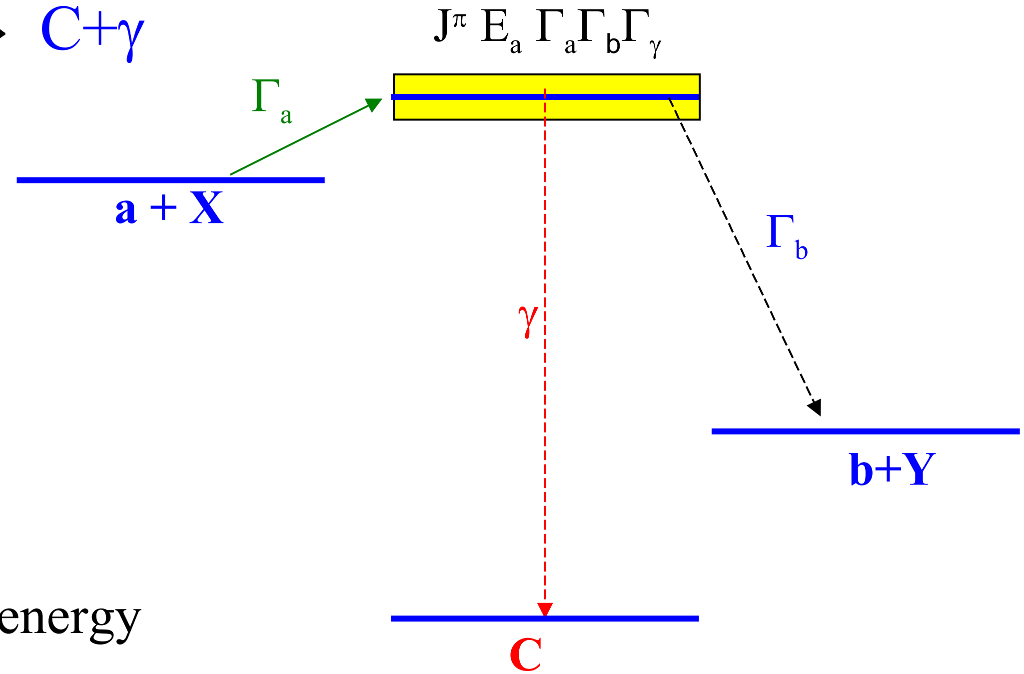
v being the relative velocity and $Y_l^m(\theta,\Phi)$ the spherical harmonics
With the penetration factor for the Coulomb and centrifugal barriers:

$$P_l(E,R_N) = \frac{|\phi_l(\infty)|^2}{|\phi_l(R_N)|^2} \Rightarrow \Gamma_a = \hbar\sqrt{\frac{2E}{\mu}} P_l(E,R_N)|\phi_l(R_N)|^2$$

$|\Phi_l(R_N)|^2 = |\phi_l(R_N)/R_N|^2$ being the probability density for the appearance of the particle a at the nuclear radius R_N

Consider reaction: $a+X \rightarrow C^* \rightarrow b+Y$

$\rightarrow C+\gamma$



<u>Resonance parameters:</u>

Resonance energy: $E_r = E_x - Q$

▯ excitation energy

Partial widths:

Γ_a : Probability of the formation of the compound nucleus C^* from the entrance channel $a+X$

Γ_b : Probability of the decay of the compound state C^* to the exit channel $b+Y$

Γ_γ : Probability of the γ decay of the compound state C^* to its ground state

Spin, parity: J^π

Nuclear reaction cross sections: Sub-threshold resonances

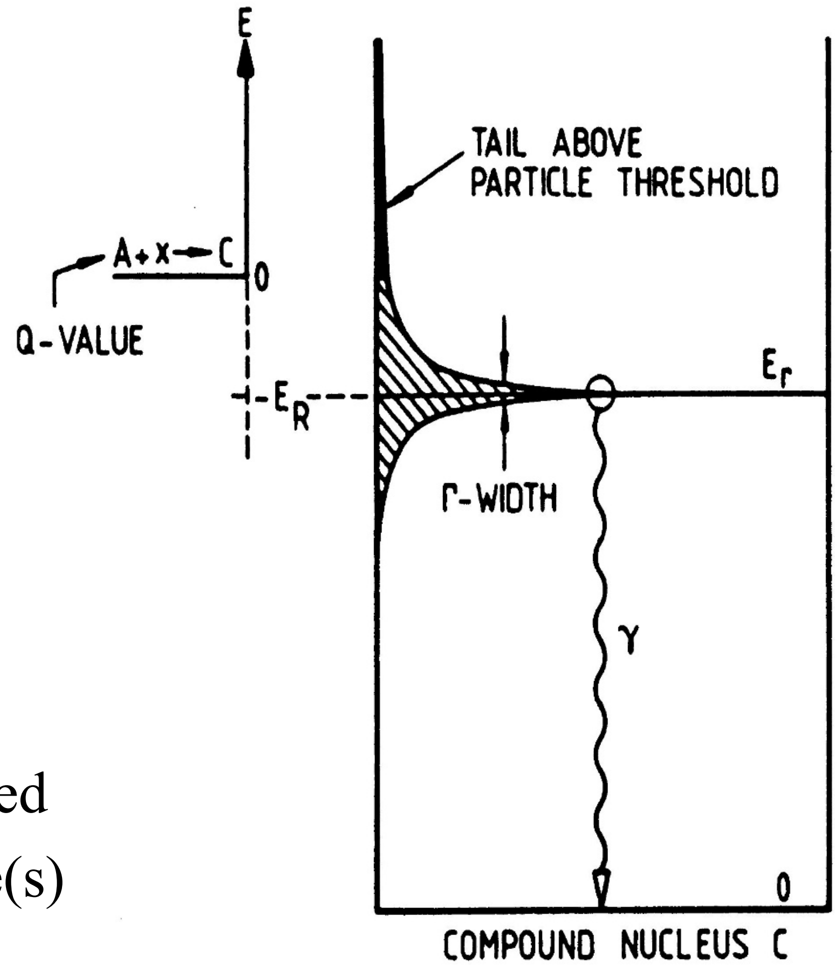
Any exited state has a finite width

$\Gamma \sim h/\tau$

high energy wing can extend
above particle threshold

⬇

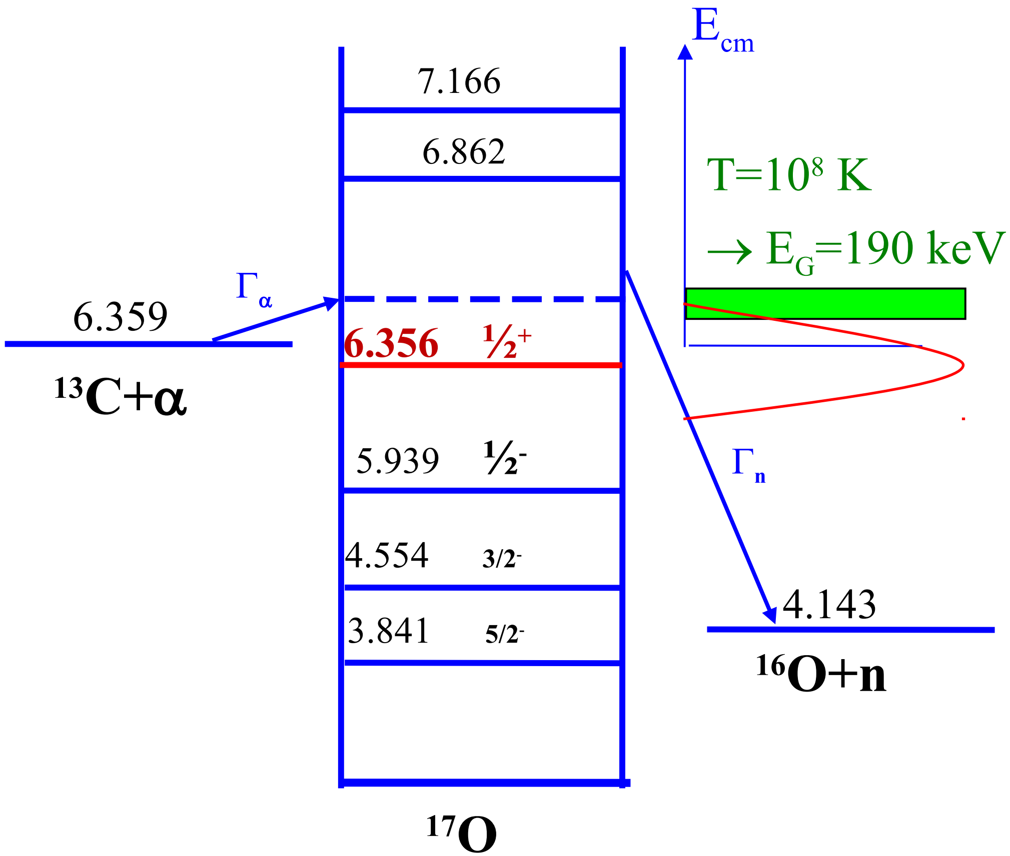
cross section can be entirely dominated
by contribution of sub-threshold state(s)



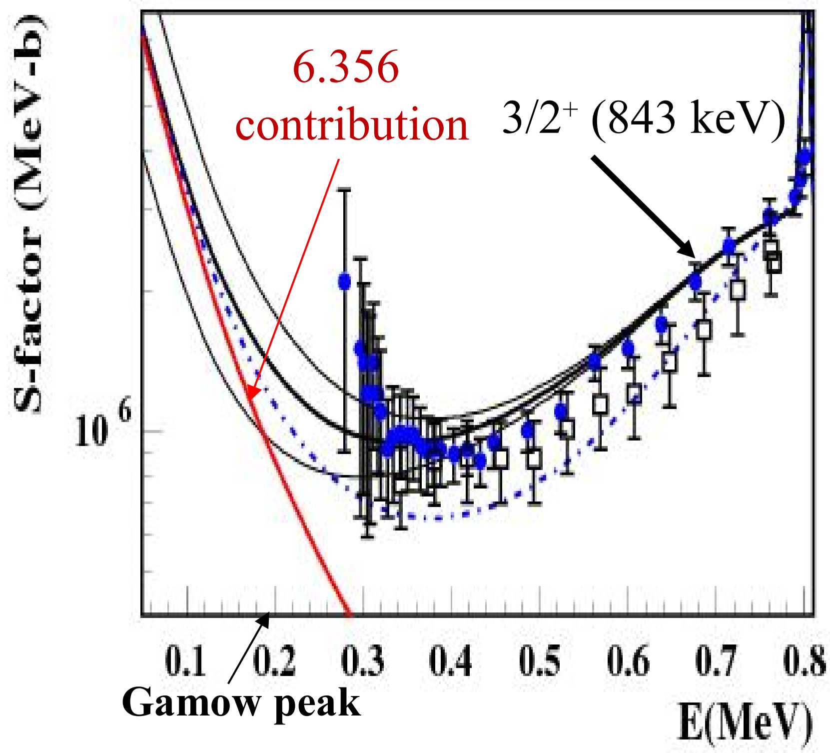
Example of sub-threshold resonant reaction: $^{13}C(\alpha,n)^{16}O$

$^{13}C(\alpha,n)^{16}O$ → main neutron source in AGB stars (1-3 $M_\odot$)

→ s-process nucleosynthesis → 90 < A < 209



See lecture III



Nuclear reaction cross sections: The Breït-Wigner cross-section

- Cross section for the resonant reaction a + X → C → Y + b, via the formation of an excited state in the compound nucleus C:

$$\sigma_{\mathrm{BW}}(E) \sim \sigma_{\max} \times f_R(E) \times \Gamma_a \Gamma_b$$

$$\boxed{\sigma_{\mathrm{BW}}(E) = \pi \lambda\!\!\!^{-2} \frac{2J+1}{(2J_a+1)(2J_X+1)} (1+\delta_{aX}) \frac{\Gamma_a \Gamma_b}{(E-E_R)^2 + (\Gamma/2)^2}}$$

where J_a and J_X are the total angular momentum of the nuclei a and X, and J that of the resonance in the compound nucleus; δ_{aX} is Kronecker's delta function

- The spin statistical factor $\boxed{\omega = (1+\delta_{aX}) \frac{2J+1}{(2J_a+1)(2J_X+1)}}$ takes into account the number of available states (selection rules)
- Note that Γ_a and Γ_b are energy dependent ($P_l(E, R_N)$…)

Nuclear reaction cross-sections: Neutron induced-reactions

A(n,x)B with $x=\gamma$, p or α

The cross section is given by :

$$\sigma_n \approx \lambda_n^2 \left| \langle B + x | H_{II} | C \rangle \langle C | H_I | A + n \rangle \right|^2 \approx \lambda_n^2 \Gamma_n(E_n) \Gamma_x(Q + E_n)$$

For thermal energies $Q \gg E_n \rightarrow \Gamma_x(Q+E_n) \approx \Gamma_x(Q)$ = constant

$\Gamma_n(E_n) \propto P_{ln}(E_n)$

for $\mathbf{l_n=0}$ $\rightarrow$ $P_0(E_n) \sim v_n$ $\Rightarrow$ $\boxed{\sigma_n(E_n) \propto \frac{1}{v_n^2} v_n = \frac{1}{v_n}}$ for non-resonant reaction

For s-wave neutrons (l=0) $\Rightarrow$ **centrifugal barrier $V_l = 0$** and also coulomb barrier $V_C = 0$

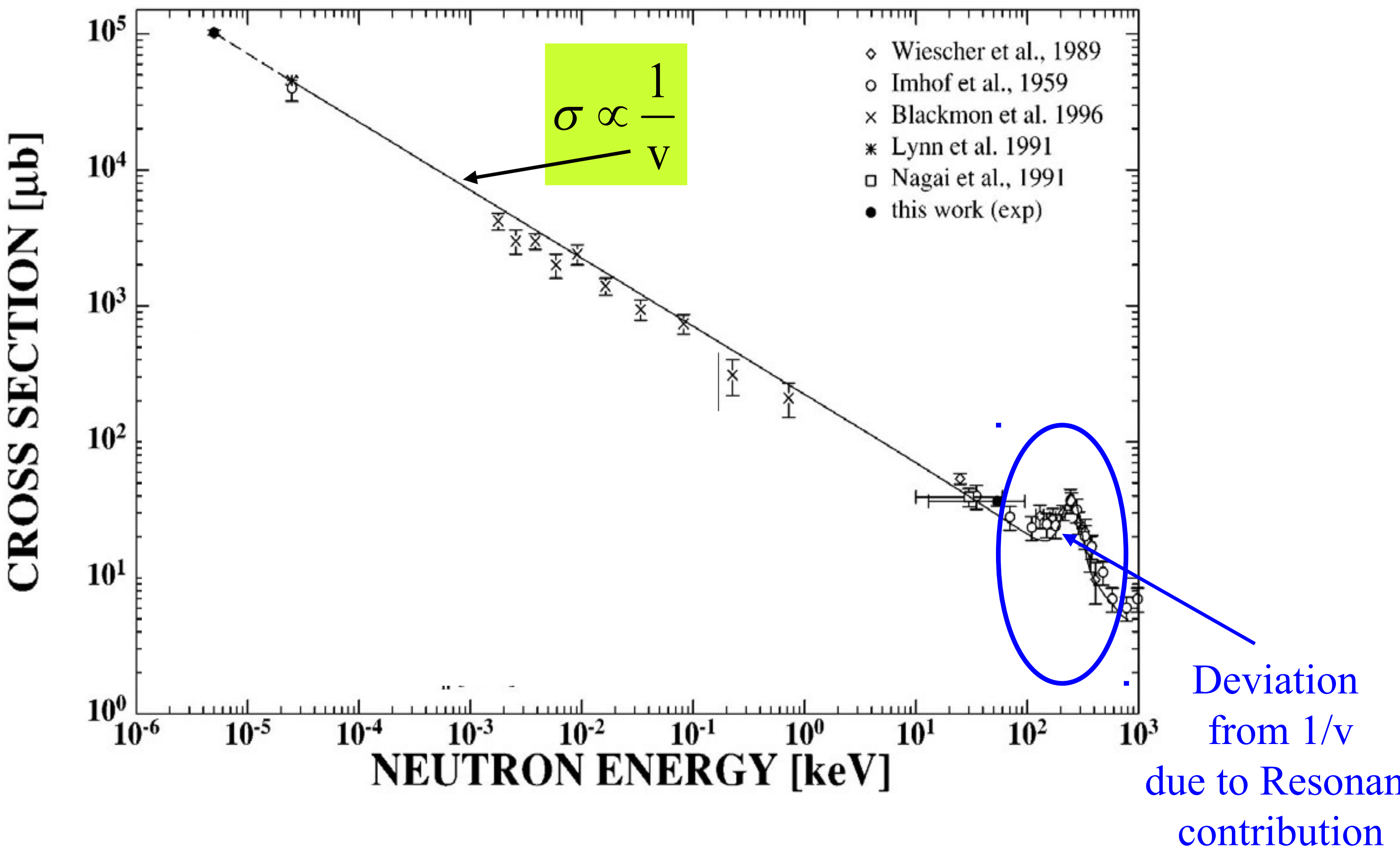
→**At low energies,** the reactions are dominated by s-wave neutron capture

→Higher *l* neutron capture plays role only at higher energies (or if l=0 capture is suppressed)

Nuclear reaction cross-sections: Neutron induced reactions

Example:

$^{7}Li(n,\gamma)^{8}Li$

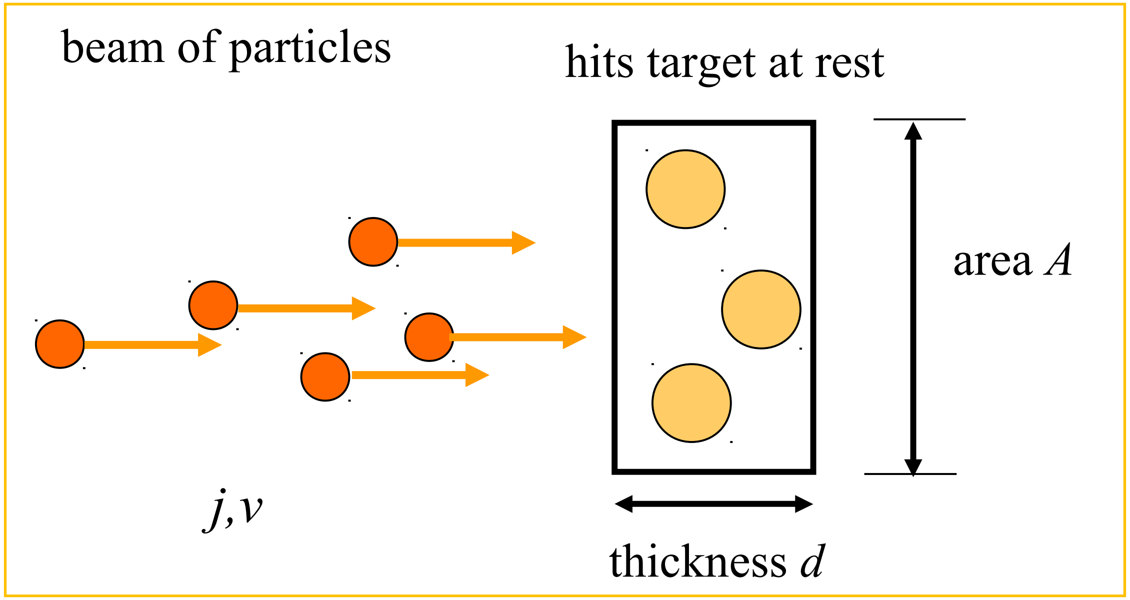


Thermonuclear reaction rates

Reaction Rates in the laboratory: Definition

<u>Reaction rate</u> a + X → b + Y

$$\sigma(\text{cm}^2) = \frac{\text{Number of reactions/unit time/nucleus X}}{\text{Number of incident particles/cm}^2\text{/unit time}}$$



assume thin target (unattenuated Intensity through target

Total reaction rate (reactions per second)

$$\boxed{R = \sigma\ I d n_T}$$

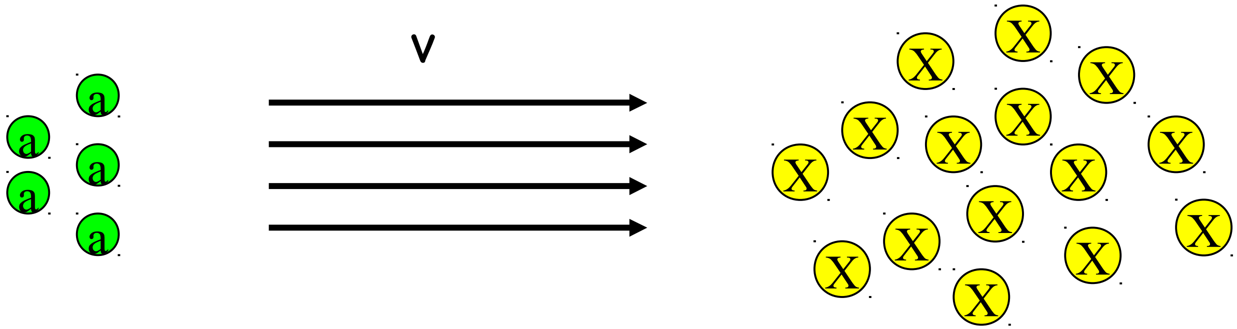
with $\boldsymbol{n_T}$: number density of target nuclei

$\boldsymbol{I = jA}$: beam number current (number of particles per second hitting the target)

note: $\boldsymbol{dn_T}$ is number of target nuclei per cm^2.

Reaction Rates in stellar environment: Definition

Mix of (fully ionized) projectiles a and target nuclei X at a temperature T



For a given relative velocity v in volume V with projectile number density N_a

The number of reactions/s $R = \sigma N_a v N_x V$

so for reaction rate per second per cm³: $\mathbf{r_{aX} = N_a N_x \sigma(v) v}$

This is proportional to the number of a-X pairs in the volume.
If projectile and target are identical, one has to divide by 2 to avoid double counting

$$r_{aX} = \frac{1}{1+\delta_{aX}} N_a N_X \sigma(v) v$$

Thermonuclear Reaction Rates: Definitions

In stellar plasma: velocity of particles varies over wide range

Reaction rate per particle pair: $$<\sigma v>_{aX} = \int_0^\infty v\sigma(v)\phi(v)dv$$ $\phi(v)$ velocity distribution

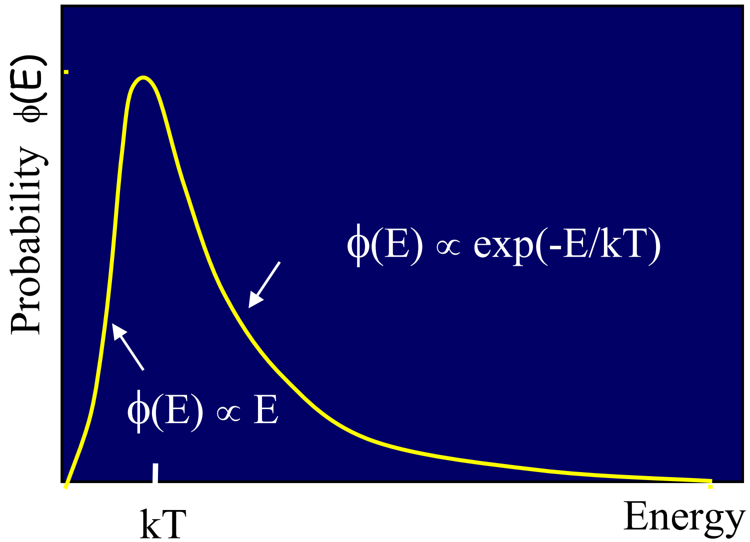
Quiescent stellar burning: non-relativistic, non-degenerate gas in thermodynamic equilibrium at temperature T

Maxwell-Boltzmann distribution:

$$\phi(v)dv = \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2kT}\right) 4\pi v^2 dv$$

μ = reduced mass

v = relative velocity



$$<\sigma v>_{aX} = \left(\frac{8}{\pi\mu_{aX}}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma(E) \exp\left(-\frac{E}{kT}\right) E\, dE$$

Thermonuclear Reaction Rates: Definitions

Total reaction rate $R_{aX} = \frac{1}{1+\delta_{aX}} \mathrm{N_a N_X} \langle \sigma v \rangle_{aX}$ reactions cm^{-3} s^{-1}

$\mathrm{N_i}$ = number density

expressed in terms abundances

$$R_{aX} = \frac{1}{1+\delta_{aX}} Y_X Y_a \rho^2 \mathrm{N_A^2} < \sigma v >$$ reactions per s and cm^3

$$\lambda = \frac{1}{1+\delta_{aX}} Y_a \rho \underbrace{\mathrm{N_A} < \sigma v >}$$ reactions per s and target nucleus

this is usually referred to
as the **stellar reaction rate**
of a specific reaction

$\mathrm{N_A}$ = Avogadro number

units of stellar reaction rate $\mathrm{N_A} < \sigma v >$: usually cm^3/mole/s

Lets assume the only reaction that involves nuclei X and Y is destruction (production) of X (Y) by X capturing the projectile a:

$$\mathrm{X} + \mathrm{a} \rightarrow \mathrm{Y}$$

The reaction is a random process with const probability (as long as the conditions are unchanged) and therefore governed by the same laws as radioactive decay:

$$\frac{dn_X}{dt} = -n_X \lambda = -n_X Y_a \rho \mathrm{N_A} < \sigma v >$$

$$\frac{dn_Y}{dt} = +n_X \lambda$$

consequently:

$$n_X(t) = n_{0X} e^{-\lambda t}$$

$$n_Y(t) = n_{0X}\left(1 - e^{-\lambda t}\right)$$

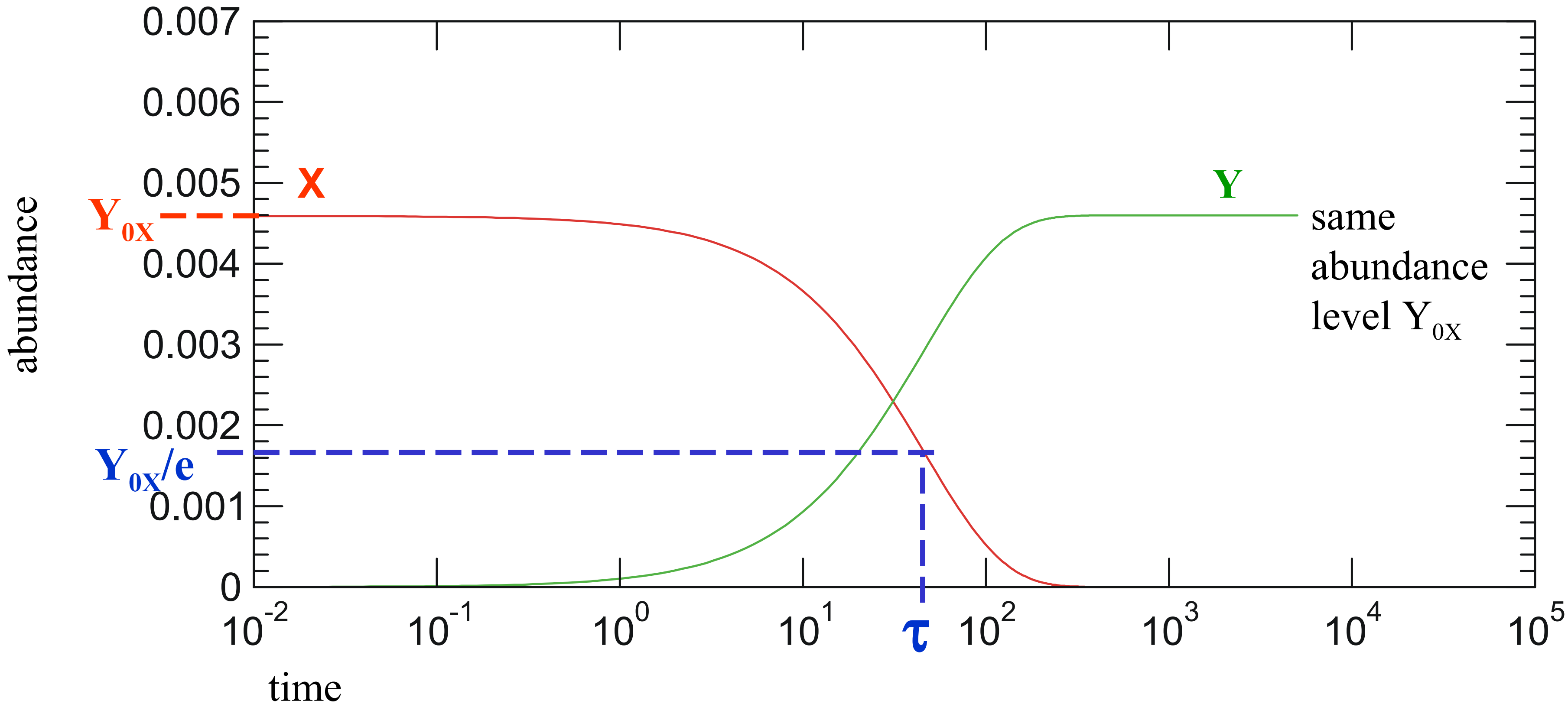
and of course

$$Y_X(t) = Y_{0X} e^{-\lambda t}$$

$$Y_Y(t) = Y_{0X}\left(1 - e^{-\lambda t}\right)$$

after some time, nucleus X is entirely converted to nucleus Y

Example:



Lifetime of X (against destruction via the reaction X+a) : $\tau = \frac{1}{\lambda} = \frac{1}{Y_a \rho \mathrm{N_A} <\sigma v>}$

(of course half-life of A $T_{1/2} = \ln 2/\lambda$)

Thermonuclear Reaction Rates: Energy generation

Consider the reaction X+a $\rightarrow$ Y

Reaction Q-value: Energy generated (if >0) by a single reaction:

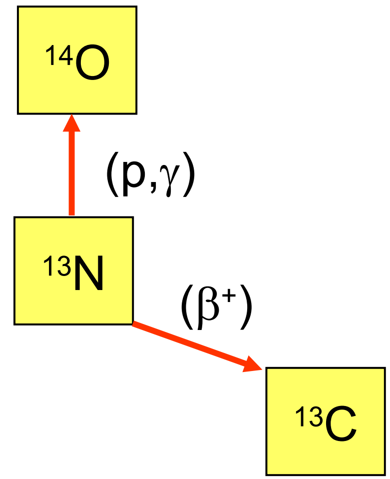
$$Q = c^2 \left(\sum_{\text{initial nuclei i}} m_i - \sum_{\text{final nuclei j}} m_j \right)$$

Energy generation: Energy generated per g and second by a reaction:

$$\varepsilon = \frac{rQ}{\rho} = Q \frac{1}{1+\delta_{aX}} Y_X Y_a \rho \mathrm{N}_{\mathrm{A}}^2 < \sigma v >$$

Thermonuclear Reaction Rates: various reactions destroying a nucleus

Example: in the CNO cycle, ^{13}N can either capture a proton or β decay.



each destructive reaction i has a rate λ_i

Total lifetime

the total destruction rate for the nucleus is then $$\lambda = \sum_i \lambda_i$$

its total lifetime $$\tau = \frac{1}{\lambda} = \frac{1}{\sum_i \lambda_i}$$

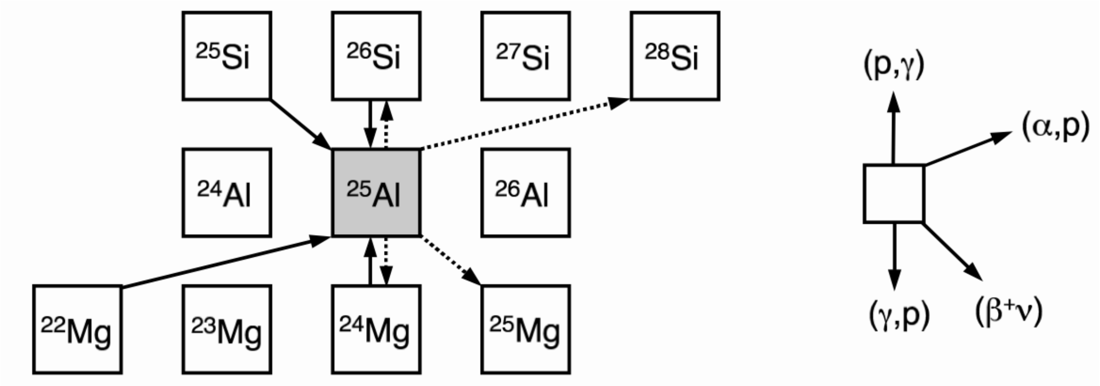
Branching

the reaction flow branching into reaction i, $\boldsymbol{b_i}$ is the fraction of destructive flow through reaction i. (or the fraction of nuclei destroyed via reaction i)

$$b_i = \frac{\lambda_i}{\sum_j \lambda_j}$$

The nucleosynthesis equations

- Evolution of the densities: system of coupled differential equations (solved numerically)



→ **Nuclear reaction network**

$$
\begin{aligned}
\frac{d(N_{^{25}\mathrm{Al}})}{dt} =& \left.\begin{aligned} & N_{\mathrm{H}} N_{^{24}\mathrm{Mg}} \langle \sigma v \rangle_{^{24}\mathrm{Mg(p,}\gamma)} + N_{^{4}\mathrm{He}} N_{^{22}\mathrm{Mg}} \langle \sigma v \rangle_{^{22}\mathrm{Mg}(\alpha,\mathrm{p})} \\ & + N_{^{25}\mathrm{Si}} \lambda_{^{25}\mathrm{Si}(\beta^+\nu)} + N_{^{26}\mathrm{Si}} \lambda_{^{26}\mathrm{Si}(\gamma,\mathrm{p})} + \ldots \end{aligned}\right\} \text{production} \\
& \left.\begin{aligned} & - N_{\mathrm{H}} N_{^{25}\mathrm{Al}} \langle \sigma v \rangle_{^{25}\mathrm{Al(p,}\gamma)} - N_{^{4}\mathrm{He}} N_{^{25}\mathrm{Al}} \langle \sigma v \rangle_{^{25}\mathrm{Al}(\alpha,\mathrm{p})} \\ & - N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\beta^+\nu)} - N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\gamma,\mathrm{p})} - \ldots \end{aligned}\right\} \text{destruction}
\end{aligned}
$$

- Nuclear energy production rate:

$$\varepsilon = \sum_{ijk} \frac{N_i N_j}{1+\delta_{ij}} \langle \sigma v \rangle_{ijk} Q_{ijk}$$

where Q_{ijk} is the Q-value of the reaction $i+j \rightarrow k$