Dense Stellar Matter in Strong Quantizing Magnetic Field- An Overview

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The Subject Matter is Related with the Strongly Degenerate Neutron Star Matter

Thermonuclear Reactions:

$\boxed{H \to He} \Longrightarrow \boxed{He \to C, O} \Longrightarrow \boxed{C \to Ne, Mg} \Longrightarrow \boxed{Ne \to Mg, Si} \Longrightarrow \boxed{O \to Si, S, Ar, Ca} \Longrightarrow \boxed{Si \to Fe, Ni} \Longrightarrow \boxed{Fe \to \text{SN-Explosion}}$.

Iron is the most stable nucleus - the thermo-nuclear reaction at the core come to a halt- core mass increases- the density and temperature also rise- the core becomes unstable.

The stability sets in when the core density reaches $\sim 10^9$gm/cc - process of neutronization begins- the capture of electrons by iron nuclei - simultaneous conversion of protons into neutrons and emission of neutrinos or photo-disintegration of iron nuclei.

The central pressure decreases, core collapses- the neutrinos escaping from the central core are absorbed by the outer layers - additional transmission of energy - results in the expulsion of envelope, or in other wards, triggers a supernova explosion.

The remaining core becomes a neutron star surrounded by the cloud of ejected matter

The energy released in photo-disintegration of iron nuclei is so high that it results total fragmentation of the whole star.

In 1934 Baade and Zwicky proposed the idea of neutron stars. The radius of a neutron star is very small ($\approx$ 10km), mass is $\sim$ a few times $M_{\odot}$, density of matter inside is very high and more strongly bound by gravitational force than the ordinary stars.

In a typical neutron star, the central density $> 10^{14}$gm/cc, and almost all the electrons have combined with protons to make neutrons; $n_e = n_p \approx 0.03 n_n$- hence the name neutron star

Gravitational contraction is balanced by the degenerate pressure of neutron gas.

Matter in β-equilibrium: $\mu_n = \mu_p + \mu_e$

At still higher densities- electron Fermi momentum $>$ muon rest mass- energetically favorable to produce μ^- instead of e^-- charge neutrality condition: $n_p = n_e + n_\mu$.

At Further higher densities- more and more massive hyperons are produced in dense neutron star matter.

High density- superfluity neutron matter- droplets of superconducting proton matter. A bit exotic phenomena, e.g., pion and kaon condensation are also expected in such dense neutron star matter.

In the extreme case- if the density $>> n_0$ formation of quark matter- quark star, strange star or hybrid star.

Magnetized Objects in Nature

A Comparative Study:

a) The Earths magnetic field measured at the N magnetic pole 0.6G.

b) A common, hand-held magnet 100G.

c) The magnetic field in strong sunspots 4000G.

d) The strongest, sustained (i.e., steady) magnetic fields achieved so far in the laboratory generated by huge electromagnets 4.5×10^5G.

e) The strongest man-made fields ever achieved, if only briefly made using focused explosive charges; lasted only $4 - 8$ microseconds. 10^7G.

f) The strongest fields ever detected on non-neutron stars $\rightarrow$ strongly-magnetized compact white dwarfs $\sim 10^8$G.

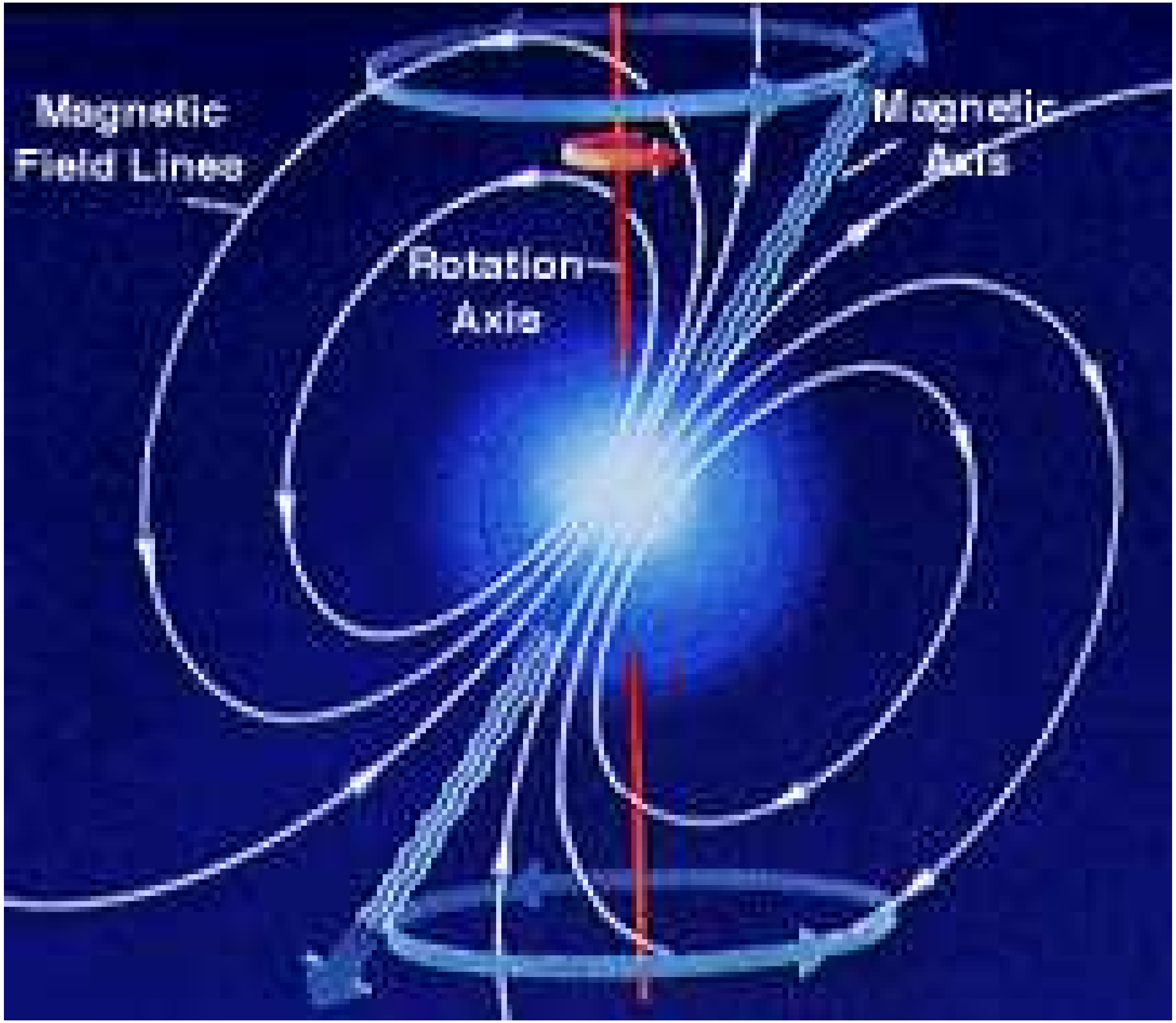
g) Typical surface & polar magnetic fields of radio pulsars the most familiar kind of neutron star; more than a thousand are known to astronomers $10^{12} - 10^{13}$G.

h) Milli-second Pulsars: Old neutron stars- Magnetic field is very low- $\sim 10^{7-8}$G.

i) Magnetars soft gamma repeaters (SGR and anomalous X-ray pulsars AXP (These are surface, polar fields. Magnetar interior fields may range up to 10^{16}G, with field lines probably wrapped in a toroidal geometry inside the star.) $10^{14} - 10^{15}$G.

Physicists have not made steady fields stronger than 4.5×10^5 Gauss in the lab because the magnetic stresses of such fields exceed the tensile strength of terrestrial materials.

Magnetic Field Structure of Magnetars/Neutron Stars/Pulsars



Origin of Neutron Star Strong Magnetic Field

Flux conservation in the case of conventional radio pulsars.

Dynamo Mechanism: A combination of rotation and convection produces strong magnetic field in the case of magnetars.

Magnetic Field Affects:

$\Downarrow$

1. Equation of State

2. Properties of NS's (M-R relation etc.)

3. Phase transition to quark matter in NS

4. Surface/crustal properties

5. Weak and Electromagnetic processes

6. NS cooling – Thermal evolution

7. Kinetic coefficients (Shear and bulk viscosity coefficients, heat conductivity and electrical conductivity)

8. Evolution of magnetic field

9. Structural deformation of NS in presence of ultra-strong magnetic field- Emission of gravity waves

Classical GTR $\longrightarrow$ Oblate shape
QM calculation $\longrightarrow$ Prolate shape
Extreme case $\Longrightarrow$ Black disk or black string.

$$\Downarrow$$

Some Unique Type Effects:

1. Complex nature of nucleon mass

2. Peculiar behaviour of electron gas- Neutral super-fluid

3. Deformation of atoms and hadrons

4. Photon splitting

5. Chiral symmetry violation

Atoms in very strong magnetic fields

The strongest magnetic field that you are ever likely to encounter personally is about 10^4 Gauss if you have Magnetic Resonance Imaging (MRI) scan for medical diagnosis. Such fields pose no threat to your health, hardly affecting the atoms in your body. Fields in excess of 10^9 Gauss, however, would be instantly lethal. Such fields strongly distort atoms, compressing atomic electron clouds into cigar shapes, with the long axis aligned with the field, thus rendering the chemistry of life impossible. A magnetar within $\leq 10^6$ Km. would thus kill you via pure static magnetism - if it didn't already get you with X-rays, gamma rays, high energy particles, extreme gravity, bursts and flares.

In fields much stronger than 10^9 Gauss, atoms are compressed into thin needles. At 10^{14} Gauss, atomic needles have widths of about 1% of their length, hundreds of times thinner than unmagnetized atoms. Such atoms can form polymer-like molecular chains or fibres. A carpet of such magnetized fibber's probably exist at the surface of a magnetar, at least in places where the surface is cool enough to form atoms.

Charged Particles in Ultra-strong magnetic field

Consider electron in ultra-strong magnetic field. If Cyclotron Quantum $\geq$ Rest Mass Energy - A new effect- the Quantum Mechanical effect of strong magnetic field- Landau Diamagnetism. For electron the critical field $= 4.4 \times 10^{13}$ Gauss. (This field-strength given by a combination of fundamental constants:

$$B_Q = \frac{m_e^2 c^3}{\hbar e},$$

m_e- rest mass of the electron, c- the speed of light, $\hbar$- reduced Planck's constant, and e- the magnitude of charge on an electron.)

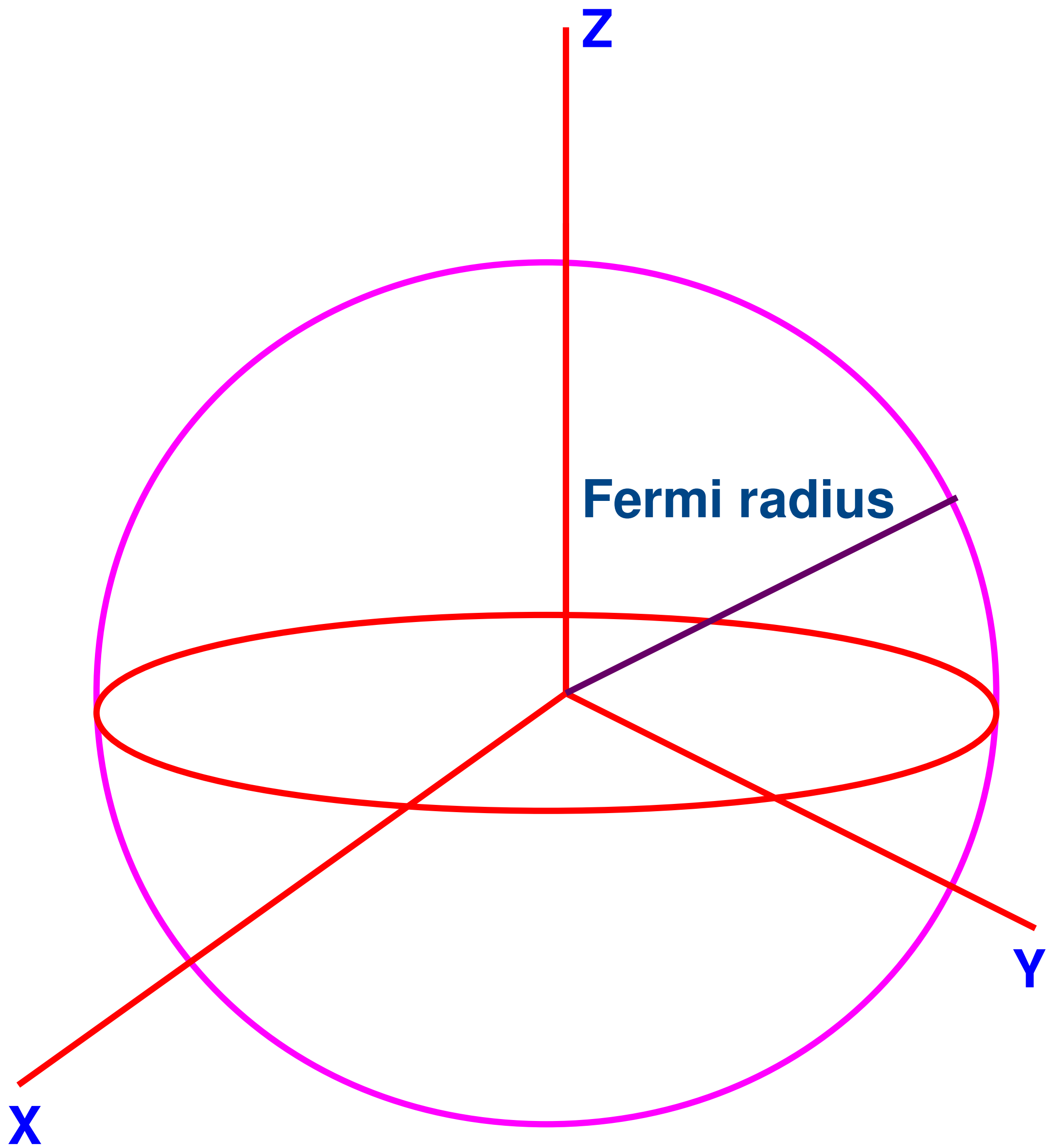
If $B \geq B_Q$ phase space - cylindrical- momentum along B- continuous $(-\infty \leq p_z \leq +\infty)$, in the $\perp$-plane- momentum gets quantized- $(p_\perp = (2\nu eB)^{1/2} = (2\nu m\hbar\omega)^{1/2})$- Landau quantization, $(\nu = 0, 1, 2, ..)$-

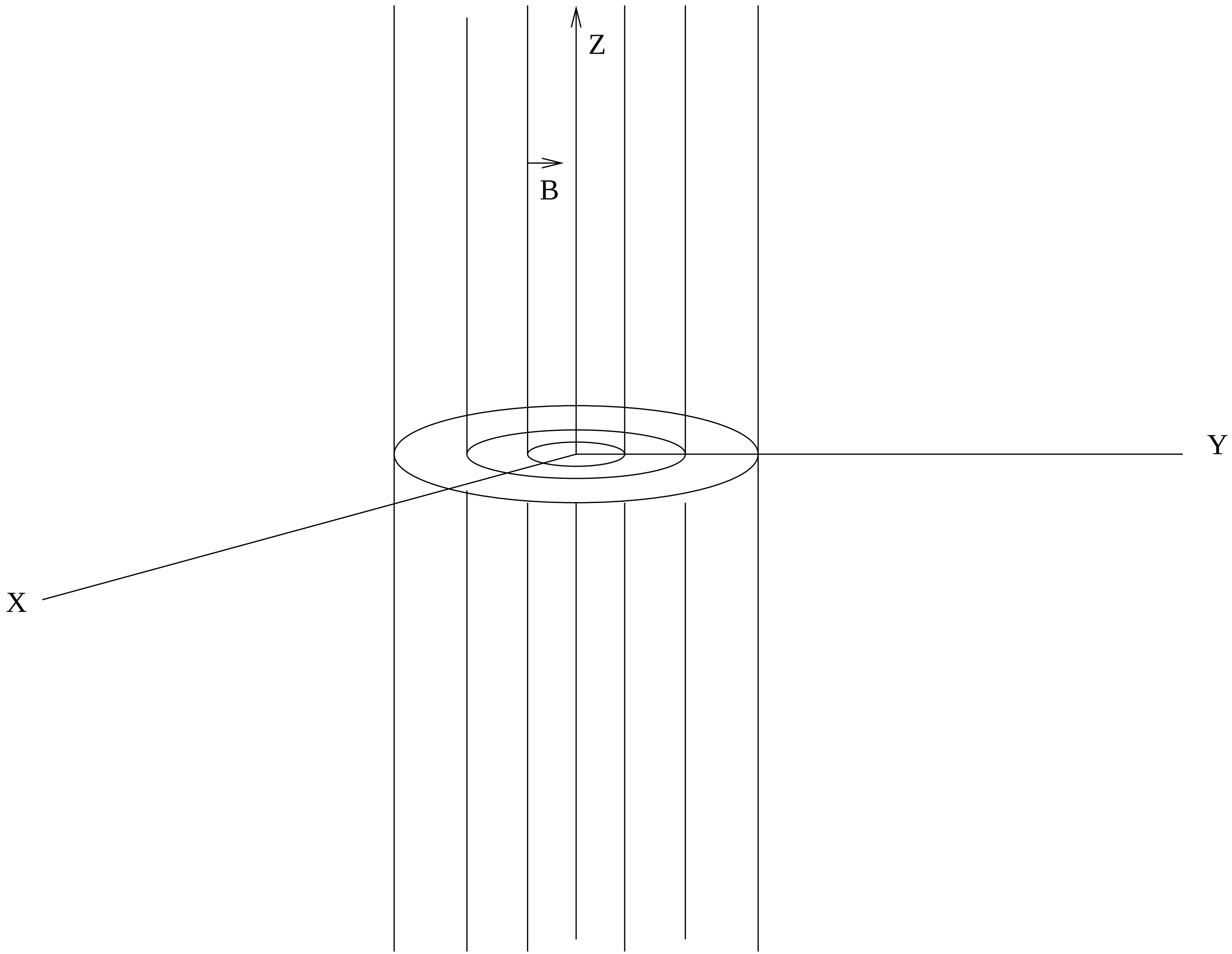
Energy Eigen Value: (NR)

$$E_\nu = \frac{p_z^2}{2m} + (\nu + \frac{1}{2})\hbar\omega = \frac{p_z^2}{2m} + \frac{p_\perp^2}{2m}, \;\; \omega = \frac{eB}{2mc}$$

Energy Eigen Value: (R)

$$E_\nu = (p_z^2 c^2 + m^2 c^4 + 2\nu\hbar ceB)^{1/2}$$


Z
Fermi radius
Y
X



Z
B
Y
X

Further, the phase space volume integral in the momentum space in this quantized condition is given by

$$\frac{1}{(2\pi)^3}\int d^3p f(p) = \frac{1}{(2\pi)^3}\int dp_z d^2p_\perp f(p) = \frac{eB}{4\pi^2}\sum_{\nu=0}^{\nu=\infty}(2-\delta_{\nu 0})\int_{-\infty}^{+\infty} dp_z f(\nu, p_z)$$

Factor $2-\delta_{\nu 0}$- the zeroth Landau level is singly degenerate, whereas all other states are doubly degenerate.

Quantum Mechanical Equation of Charged Particle in Strong Quantizing Magnetic Field:

Non-Relativistic: (Schrödinger equation)

$$\frac{d^2F}{dx^2} + \frac{2m}{\hbar^2}\left[E - \frac{p_z^2}{2m} - \frac{e^2B^2}{2mc^2}(x - x_0)\right]F(x) = 0$$

where $\vec{A} \equiv (0, xB, 0)$, $x_0 = cp_y/eB$ and

$$\psi(x, y, z) = N \exp\left[\frac{i}{\hbar}(p_y y + p_z z)\right]F(x)$$

Relativistic: (Dirac equation)

$$[\gamma_\mu(i\partial^\mu - eA^\mu) - m]\psi(x) = 0$$

The modified form of spinor solutions of Dirac equation:

$$\psi(x) = \frac{1}{(L_y L_z)^{1/2}} \exp\{-iE_\nu t + ip_y y + ip_z z\} u^{\uparrow\downarrow}(x)$$

where

$$u^{\uparrow}(x) = \frac{1}{[2E_\nu(E_\nu + m)]^{1/2}} \begin{pmatrix} (E_\nu + m) I_{\nu;p_y}(x) \\ 0 \\ p_z I_{\nu;p_y}(x) \\ -i(2\nu eB)^{1/2} I_{\nu-1;p_y}(x) \end{pmatrix}$$

and

$$u^{\downarrow}(x) = \frac{1}{[2E_\nu(E_\nu + m)]^{1/2}} \begin{pmatrix} 0 \\ (E_\nu + m) I_{\nu-1;p_y}(x) \\ i(2\nu eB)^{1/2} I_{\nu;p_y}(x) \\ -p_z I_{\nu-1;p_y}(x) \end{pmatrix}$$

$\uparrow$ and $\downarrow$ $\Rightarrow$ up and down spin states respectively,

$$I_\nu = \left(\frac{qB}{\pi}\right)^{1/4} \frac{1}{(\nu!)^{1/2}} 2^{-\nu/2} \exp\left[-\frac{1}{2} eB \left(x - \frac{p_y}{eB}\right)^2\right] H_\nu\left[(eB)^{1/2}\left(x - \frac{p_y}{eB}\right)\right]$$

H_ν -Hermite polynomial of order ν, L_y, L_z- length scales along Y and Z directions respectively. Here we have taken $\hbar = k_B = c = 1$.

Stability of Matter

Atoms, Nuclei, Electron gas at the crust of NS, Neutron star matter and Quark Matter become more stable in presence of B.

Non-interacting charge neutral degenerate $n - p - e$ matter in β equilibrium with $B \neq 0$:

URCA Processes:

$n \longrightarrow p + e^- + \bar{\nu}_e$

$p + e^- \longrightarrow n + \bar{\nu}_e$

$$
\begin{aligned}
n_p &= n_e \\
\mu_n &= \mu_p + \mu_e \\
n &= n_p + n_n
\end{aligned}
$$

Number Density

$$n_i = \frac{eB}{\pi^2} \sum_{\nu=0}^{[\nu_{\rm max}^{(i)}]} (2 - \delta_{\nu 0}) p_{F_i}$$

$i = p$ or e

For neutron

$$n_n = \frac{1}{6\pi^2} p_{F_n}^3$$

The expressions for degenerate pressure and energy density:

$$\begin{aligned} P_i &= -\Omega_{i,V} \\ &= \frac{q_i g_i B}{2\pi^2} \sum_{\nu=0}^{[\nu_{max}^{(i)}]} [\frac{1}{2}\mu_i (\mu_i^2 - M_\nu^{(i)2})^{1/2} \\ &- \frac{1}{2} M_\nu^{(i)2} \ln \left\{ \frac{\mu_i + (\mu_i^2 - M_\nu^{(i)2})^{1/2}}{M_\nu^{(i)}} \right\}] \end{aligned}$$

and

$$
\begin{aligned}
\epsilon_i &= \Omega_{i,V} + \mu_i n_i \\
&= \frac{q_i g_i B}{2\pi^2} \sum_{\nu=0}^{[\nu_{max}]} \Big[\frac{1}{2}\mu_i(\mu_i^2 - M_\nu^{(i)2})^{1/2} \\
&+ \frac{1}{2} M_\nu^{(i)2} \ln \left\{ \frac{\mu_i + (\mu_i^2 - M_\nu^{(i)2})^{1/2}}{M_\nu^{(i)}} \right\} \Big]
\end{aligned}
$$

where μ_i chemical potential $M_\nu^{(i)} = (m_i^2 + 2\nu q_i B)^{1/2}$.

Whereas for neutron they are:

$$
\begin{aligned}
P_n &= \frac{1}{8\pi^2}[2\mu_n(\mu_n^2 - m_n^2)^{3/2} - 3m_n^2\mu_n(\mu_n^2 - m_n^2)^{1/2} \\
&+ 3m_n^4 \ln \left\{ \frac{\mu_n + (\mu_n^2 - m_n^2)^{1/2}}{m_n} \right\}]
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon_n &= \frac{3}{8\pi^2}[2\mu_n^3(\mu_n^2 - m_n^2)^{1/2} - m_n^2\mu_n(\mu_n^2 - m_n^2)^{1/2} \\
&- m_n^4 \ln \left\{ \frac{\mu_n + (\mu_n^2 - m_n^2)^{1/2}}{m_n} \right\}]
\end{aligned}
$$

Considering β-equilibrium and charge neutrality one can see that

$$\frac{\epsilon/n_B|_{B>B_c}}{\epsilon/n_B|_{B<B_c}} \ll 1$$

Stability of strange quark matter at $T = 0$- Witten $\longrightarrow$ Quark matter in B -More stable.

Formalism in brief:

u, d, s and e in β-equilibrium and charge neutral:

Now for $B > B^{(c)}$, with the relation $p_F^2 \geq 0$, the maximum value of Landau quantum number at zero temperature is given by

$$\nu_{\text{max}} = \left[\frac{(\mu^2 - m^2)}{2eB}\right]$$

which is an integer but less than the actual value of the quantity within the third brackets at the right hand side and μ is the chemical potential of the charged particle. The upper limit of ν-sum will now be ν_{max} instead of ∞.

The direct URCA process:

$$d \to u + e^- + \bar{\nu}_e, \quad s \to u + e^- + \bar{\nu}_e$$

the reverse process

$$u + e^- \to d + \nu_e. \;\; u + e^- \to s + \nu_e$$

The system is also charge neutral

$$2n_u - n_d - n_s - 3n_e = 0$$

in β-equilibrium

$$\mu_d = \mu_s \quad \text{and} \quad \mu_d = \mu_u + \mu_e$$

The number density for ith species is given by

$$n_i = \frac{eB}{2\pi^2} \sum_{\nu=0}^{\nu_{\mathrm{max(i)}}} (2 - \delta_{\nu 0}) p_{F_i}$$

Thermodynamic potential:

$$\begin{aligned} \Omega &= -T \ln Z \\ &= \sum_i \Big[\Omega_{i,V}(T,\mu_i)V + \Omega_{i,S}(T,\mu_i)S + \Omega_{i,C}(T,\mu_i)C \Big] + B_P V \end{aligned}$$

Sum is over u, d, s-quarks and electron (e).

First term is the volume contribution:

$$\Omega_{i,V}(T,\mu_i) = -\frac{Tg_i}{(2\pi)^3}\int d^3k \ln\left(1+\exp(\beta(\mu_i-\epsilon_i))\right)$$

g_i is the degeneracy of the i-th species ($= 6$ for quarks and 2 for electron) and V is the volume occupied by the system.

Second term is the surface contribution:

$$\Omega_{i,S}(T,\mu_i) = \frac{g_i T}{64\pi^2}\int \frac{d^3k}{|\vec{k}|}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{k}{m_i}\right)\right] \ln\left(1+\exp(\beta(\mu_i-\epsilon_i))\right) = \sigma$$

S is the area of the surface enclosing the volume V and σ is the surface energy per unit area or the surface tension of quark matter.

The last term corresponds to curvature correction:

$$\Omega_{i,C}(T,\mu_i) = \frac{Tg_i}{48\pi^3}\int \frac{d^3k}{|\vec{k}|^2}\ln\left[1+\exp(\beta(\mu_i-\epsilon_i))\right] \left[1-\frac{3}{2}\frac{k}{m_i}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{k}{m_i}\right)\right)\right]$$

where C is the length of the line element drawn on the surface S.

In presence of strong magnetic field:

$$\Omega_{i,V} = -T\frac{q_i g_i B}{2\pi^2}\sum_{\nu=0}^{\infty}\int_0^{\infty} dk_z \ln\left(1+\exp(\beta(\mu_i-\epsilon_i^{(\nu)}))\right)$$

$$\Omega_{i,S} = T\frac{q_i g_i B}{16\pi}\sum_{\nu=0}^{\infty}\int_0^{\infty}\frac{dk_z}{\sqrt{(k_z^2+k_{\perp,(i)}^2)}}\ln\left(1+\exp(\beta(\mu_i-\epsilon_i^{(\nu)}))\right)\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{k}{m_i}\right)\right]$$

and

$$\Omega_{i,C} = T\frac{q_i g_i B}{12\pi^2}\sum_{\nu=0}^{\infty}\int_0^{\infty}\frac{dk_z}{(k_z^2+k_{\perp,(i)}^2)}\ln\left(1+\exp(\beta(\mu_i-\epsilon_i^{(\nu)}))\right)\left[1-\frac{3}{2}\frac{k}{m_i}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{k}{m_i}\right)\right)\right]$$

where $k_\perp^{(i)} = 2\nu q_i B$

Surface and curvature terms play significant roles during quark bubble nucleation in dense neutron matter, and are not important for a bulk quark matter system.

Bulk Strange Quark Matter

Consider only the bulk term at $T = 0$.

Free energy density (General expression):

$$U_i = \Omega_{i,V} + \mu_i n_i - T \left(\frac{\partial \Omega_{i,V}}{\partial T} \right)_{\mu_i}$$

The last term comes from the non-zero entropy of the system, which is zero for $T = 0$. Then the total energy density of the confined SQM is given by

$$U = \sum_i U_i + B_P$$

The expressions for degenerate pressure and free energy density:

$$\begin{aligned} P_i &= -\Omega_{i,V} \\ &= \frac{q_i g_i B}{2\pi^2} \sum_{\nu=0}^{[\nu_{max}^{(i)}]} [\frac{1}{2}\mu_i(\mu_i^2 - M_\nu^{(i)2})^{1/2} \\ &- \frac{1}{2} M_\nu^{(i)2} \ln \left\{ \frac{\mu_i + (\mu_i^2 - M_\nu^{(i)2})^{1/2}}{M_\nu^{(i)}} \right\}] \end{aligned}$$

and

$$\begin{aligned}\epsilon_i &= \Omega_{i,V} + \mu_i n_i \\ &= \frac{q_i g_i B}{2\pi^2} \sum_{\nu=0}^{[\nu_{max}]} \Big[\frac{1}{2}\mu_i(\mu_i^2 - M_\nu^{(i)2})^{1/2} \\ &+ \frac{1}{2} M_\nu^{(i)2} \ln\left\{\frac{\mu_i + (\mu_i^2 - M_\nu^{(i)2})^{1/2}}{M_\nu^{(i)}}\right\}\Big]\end{aligned}$$

where $M_\nu^{(i)} = (m_i^2 + 2\nu q_i B)^{1/2}$.

Whereas for s-quark, they are:

$$\begin{aligned}P_s &= \frac{1}{8\pi^2}[2\mu_s(\mu_s^2 - m_s^2)^{3/2} - 3m_s^2\mu_s(\mu_s^2 - m_s^2)^{1/2} \\ &+ 3m_s^4 \ln\left\{\frac{\mu_s + (\mu_s^2 - m_s^2)^{1/2}}{m_s}\right\}]\end{aligned}$$

and

$$\begin{aligned}\epsilon_s &= \frac{3}{8\pi^2}[2\mu_s^3(\mu_s^2 - m_s^2)^{1/2} - m_s^2\mu_s(\mu_s^2 - m_s^2)^{1/2} \\ &- m_s^4 \ln\left\{\frac{\mu_s + (\mu_s^2 - m_s^2)^{1/2}}{m_s}\right\}]\end{aligned}$$

Considering β-equilibrium and charge neutrality one can see that

$$\frac{\epsilon/n_B|_{B>B_c}}{\epsilon/n_B|_{B<B_c}} < 10$$

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Nucleation of Quark Bubble in Compact Neutron matter

Landau & Lifshitz, Statistical Mechanics, Part-2

First order quark-hadron phase transition initiated by the nucleation of droplets of quark matter in presence of strong magnetic field.

Surface and curvature energies of the quark bubble play crucial role in droplet nucleation.

Nucleation rate of stable quark bubble in metastable neutron matter per unit volume:

$$I = I_0 \exp\left(-W_m/T\right) \approx T^4 \exp\left(-W_m/T\right)$$

where W_m is the minimum thermodynamic work to be done to create a critical quark droplet and is given by

$$W_m = \frac{4}{3}\pi\frac{\sigma^3}{(\Delta P)^2}[2 + 2(1+b)^{3/2} + 3b]$$

where $\sigma = \sigma_q + \sigma_n$ the surface tension and $\Delta P = P_q - P_n$ is the pressure difference. Here $\Delta P = P_q - P_n > 0$.

$b = 2\gamma(\Delta P)/\sigma^2$, and $\gamma = \gamma_q - \gamma_n$, stands for curvature energy density.

q- quark phase n- metastable neutron matter.

Bubble nucleation time $\tau_{\text{bubble}} \approx 10^{-23}$ sec $\approx \tau_{\text{strong}}$, the strong interaction time scale, the creation of strange quarks through weak processes within the quark droplets may be ignored (u, d-quarks only).

Temperature ($\sim 5-10$ MeV) $<<$ quark chemical potential ($\sim$ 300 MeV),-anti-quarks are ignored.

Hyperons at the core $\Longrightarrow u, d$ and s quarks in the quark bubble.

It is obvious that the surface energy diverges logarithmically in the infra red limit ($k_z \to 0$) for $\nu = 0$ (for the ground state Landau level).

To show this more explicitly:

Assume $T = 0 \quad \longrightarrow$ upper limit of the ν is ν_{max}.

Then we have writing

$$\sigma = \frac{TB_m}{16\pi}\sum_{i=u,d} g_i q_i \sum_{\nu=0}^{\infty}\int_0^{\infty}\frac{dk_z}{\sqrt{k_z^2+k_{\perp(i)}^2}}\ln\left[1+\exp\left(-\frac{\epsilon_i^{(\nu)}-\mu_i}{T}\right)\right]G$$

where

$$G = \left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{k}{m_i}\right)\right]$$

for $T \to 0$

$$\sigma(G=1) = \frac{B_m}{16\pi}\sum_{i=u,d} g_i q_i \{\sum_{\nu=0}^{\nu_{\max}^{(i)}}\int_0^{k_{F_i}}\frac{k_z dk_z}{\sqrt{k_z^2+m_i^2+k_{\perp,(i)}^2}}\ln(k_z+\sqrt{k_z^2+k_{\perp,(i)}^2})$$

$$- \sum_{\nu=0}^{\nu_{\max}} \ln(k_{\perp,(i)})(\mu_i - \sqrt{k_{\perp,(i)}^2+m_i^2})\}$$

Since $\mu_i > \sqrt{k_{\perp,(i)}^2+m_i^2}$, therefore, for $\nu = 0$ the second term becomes $+\infty$. The other part of integral, evaluated numerically, found to be a finite number.

Similarly the integral with the second part of G (which is $\neq 1$) has also been obtained numerically and is found to be finite.

Therefore the diverging property of $\sigma(G = 1)$ for $\nu = 0$ is also true for the whole surface energy / area.

Although we have assumed here $T \rightarrow 0$, this important conclusion is equally valid for any finite T.

The nucleation rate of droplet formation becomes zero, i.e., there can not be a single quark droplet formation from metastable neutron matter at the core region, if the magnetic field is of the order of or greater than the corresponding critical value.

Curvature term:

$$\gamma = \frac{T}{12\pi^2} \sum_{i=u,d} g_i q_i \sum_{\nu=0}^{\infty} \int_0^{\infty} \frac{dk_z}{k_z^2 + k_{\perp,(i)}^2} \ln\left(1 + \exp(-\beta(\epsilon_i - \mu_i))\right) G$$

where

$$G = 1 - \frac{3}{2}\frac{k}{m_i}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{k}{m_i}\right)\right)$$

Term by term evaluation:

$$I_1 = \int_0^{\infty} \frac{dk_z}{k_z^2 + k_{\perp,(i)}^2} \ln\left(1 + \exp(-\beta(\epsilon_i - \mu_i))\right)$$

Diverges in the infra red limit for $\nu = 0$. Here the divergence is $1/k_z$ type.

Explicitly form: integrate by parts:

$$I_1 = \frac{1}{T k_{\perp,(i)}} \int_0^{\infty} \tan^{-1}\left(\frac{k_z}{k_{\perp,(i)}}\right) \frac{k_z dk_z}{(k_z^2 + k_{\perp,(i)}^2 + m_i^2)^{1/2}} \frac{1}{\exp(\beta(\epsilon_i - \mu_i)) + 1}$$

which diverges for $\nu = 0$, but the divergence is not logarithmic ($I_1 \sim 1/\nu$ as $\nu \to 0$).

Second term:

$$I_2 = -\frac{3\pi}{4m_i}\int_0^\infty \frac{dk_z}{(k_z^2+k_{\perp,(i)}^2)^{1/2}} \ln\left(1+\exp(-\beta(\epsilon_i-\mu_i))\right)$$

Integrating by parts:

$$\begin{aligned} I_2 &= \frac{3\pi}{4m_i}[\{\ln(k_{\perp,(i)})\ln\left(1+\exp(-\beta(\epsilon_i-\mu_i))\right)\} \\ &- \frac{1}{T}\int_0^\infty \ln(k_z+(k_z^2+k_{\perp,(i)}^2)^{1/2})\frac{k_z dk_z}{(k_z^2+k_{\perp,(i)}^2+m_i^2)^{1/2}} \\ &\left[\frac{1}{\exp(\beta(\epsilon_i-\mu_i))+1}\right]] \end{aligned}$$

For $\nu = 0$, the first term diverges logarithmically, but unlike the second term it becomes $-\infty$. All other terms remain finite for all values of ν in the infra red as well as ultra violet limits. Now the divergences of first two terms of this eqn. can not cancel each other. The first divergence is much faster as $\nu \to 0$ than the second one, therefore the overall divergence of curvature remains positive as $\nu \to 0$.

As a consequence, nucleation of quark bubbles in metastable neutron matter will be completely forbidden.

Interacting Neutron Star Matter in Absence of Magnetic Field

Reference

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Relativistic Mean Field Theory: σ-ω Model of Nuclear Matter

Scalar field:σ couples with baryon scalar density $\rho_s = g_\sigma \overline{\psi}\psi$.

Vector field ω^μ ($\mu = 0, 1, 2, 3$) couples with baryon four-current $j^\mu = g_\omega \overline{\psi}\gamma^\mu\psi$.
g_i with $i = \sigma$ and ω are the coupling constants.

Then we have the Lagrangian density

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}[i\gamma_\mu(\partial^\mu + ig_\omega\omega^\mu) - (n - g_\sigma\sigma)]\psi \\
&+ \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu
\end{aligned}$$

where $\omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$-vector field tensor.

EL-equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

where ϕ:$\sigma, \omega^\mu, \psi, \overline{\psi}$ $\Longrightarrow$ we have with $\partial_\mu \omega^\mu = 0$ (comes automatically since $\partial_\mu j^\mu = 0$)

$$[\Box + m_\sigma^2]\sigma = g_\sigma \overline{\psi}\psi$$
$$[\Box + m_\omega^2]\omega_\mu = g_\omega \overline{\psi}\gamma_\mu \psi$$

and finally

$$[\gamma_\mu(i\partial^\mu - g_\omega \omega^\mu) - (m - g_\sigma \sigma)]\psi(x) = 0$$

Set of equations are coupled, non-linear and hence extremely difficult to solve numerically. $\Longrightarrow$ Introduced an approximation, called mean field approximation: Matter is assumed to be static and uniform in ground state and mean fields or the mean values of the scalar and vector fields are considered:
$\sigma(x) \longrightarrow <\sigma(x)> = \sigma$ and $\omega(x) \longrightarrow <\omega(x)> = \omega$ (we are using same symbols for the mean fields). $\Longrightarrow$

$$m_\sigma^2 \sigma = g_\sigma < \overline{\psi}\psi >$$
$$m_\omega^2 \omega_0 = g_\omega < \psi^\dagger \psi >$$
$$m_\omega^2 \omega_k = g_\omega < \overline{\psi}\gamma_k \psi > i$$

With mean fields, Dirac eqn. is is given by:

$$[\gamma_\mu(i\partial^\mu - g_\omega\omega^\mu) - (m - g_\sigma\sigma)]\psi(x) = 0$$

Now σ and ω are treated as background field.

With $\psi(x) \sim \psi(k)exp(-ik.x)$, we have

$$[\gamma_\mu(k^\mu - g_\omega\omega^\mu) - (m - g_\sigma\sigma)]\psi(k) = 0$$

Define:$K^\mu = k^\mu - g_\omega\omega^\mu$ and effective baryon mass $m^* = m - g_\sigma\sigma$. Then the energy eigen value $\varepsilon(k) = k_0 = K_0 + g_\omega\omega_0$, with $K_0 = [(\vec{k} - g_\omega\vec{\omega})^2 + m^{*2}]^{1/2}$.

Spatial Component of ω-Field $= 0$

Let Γ is any operator. Define single-particle expectation value: $< \overline{\psi}|\Gamma|\psi >_{k,s,\tau}$.

Subscripts: k-momentum, s-spin and τ-isospin. Expectation value in the ground state of many nucleon system:

$$< \overline{\psi}|\Gamma|\psi > = \sum_{s,\tau} \frac{1}{(2\pi)^3} \int d^3k < \overline{\psi}|\Gamma|\psi >_{k,s,\tau} \Theta(\mu - \varepsilon(k))$$

where μ-Fermi energy $\equiv$ chemical potential (at $T = 0$).

From Dirac equation:

$$k_0\psi(k) = \gamma_0(\vec{\gamma}.\vec{k} + g_\omega\gamma_\mu\omega^\mu + m^*)\psi = H_D\psi$$

where H_D is the Dirac Hamiltonian. Consider any variable ξ, such that

$$\frac{\partial}{\partial\xi} < \psi^\dagger|H_D|\psi >_{k,s,\tau} = < \psi^\dagger \left|\frac{\partial H_D}{\partial\xi}\right| \psi >_{k,s,\tau} + k_0\frac{\partial}{\partial\xi} < \psi^\dagger\psi >_{k,s,\tau}$$

The last term on rhs is zero.

$$\rho = < \psi^\dagger\psi > = \frac{4}{(2\pi)^3}\int d^3k\Theta(\mu - \varepsilon(k))$$

Hence by $\xi \longrightarrow k^i$ and taking $E(k)$ as the single-particle eigen value, we have

$$\frac{\partial}{\partial k^i}E(k) = < \overline{\psi}|\gamma^i|\psi >_{k,s,\tau}$$

Then

$$\begin{aligned} < \overline{\psi}|\gamma^i|\psi > &= \frac{4}{(2\pi)^3}\int d^3k \left[\frac{\partial}{\partial k^i}E(k)\right]\Theta(\mu - \varepsilon(k)) \\ &= \frac{4}{(2\pi)^3}\int dk^i dk^j dk^k \left[\frac{\partial}{\partial k^i}E(k)\right]\Theta(\mu - \varepsilon(k)) \\ &= \frac{4}{(2\pi)^3}\int dk^j dk^k \int dE(k^j, k^k) \end{aligned}$$

The last integral explicitly becomes zero since at any point on the Fermi surface the energy value is the Fermi energy (rotational invariance).

Therefore, $< \overline{\psi}|\gamma^i|\psi >$, the baryon three-current in the medium vanishes identically.

Hence

$$\omega^i = \frac{g_\omega}{m_\omega^2} j^i = 0$$

Only $\omega_0 \neq 0$. Further, the single-particle energy $E(k) = (k^2 + m^{*2})^{1/2}$.

Baryon density (vector density):

$$\rho =< \psi^\dagger|\psi >= \frac{4}{(2\pi)^3} \int d^3k \Theta(\mu - \varepsilon(k)) = \frac{2k_F^3}{3\pi^2}$$

Scalar density:

Now

$$< \overline{\psi}|\psi >_{k,s,\tau} = \frac{\partial E(k)}{\partial m} = \frac{m^*}{(k^2 + m^{*2})^{1/2}}$$

Then

$$\rho_s =< \overline{\psi}|\psi >= \frac{2}{\pi^2}\int_0^{k_F} k^2 dk \frac{m^*}{(k^2 + m^{*2})^{1/2}}$$

Energy density:

$$\epsilon = - < \mathcal{L} > + < \overline{\psi}\gamma_0 k_0 \psi >$$

Pressure:

$$P =< \mathcal{L} > + \frac{1}{3} < \overline{\psi}\gamma_i k_i \psi >$$

where $i = 1, 2, 3$ Hence

$$\epsilon = \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega_0^2 + \frac{2}{\pi^2}\int_0^{k_F}(k^2 + m^{*2})^{1/2}k^2 dk$$

and

$$P = -\frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega_0^2 + \frac{1}{3}\frac{2}{\pi^2}\int_0^{k_F}\frac{k^2}{(k^2 + m^{*2})^{1/2}}k^2 dk$$

then the EOS:$P \equiv P(\epsilon)$

Role of σ and ω fields are opposite in nature: σ-decreases the energy of the system, where is ω_0 increases the energy. At a particular density, σ and ω_0 will be such that energy will be minimum $\longrightarrow$ saturation energy at saturation density. Saturation energy gives saturation binding energy. Binding energy/nucleon:

$$\frac{B}{A} = \left(\frac{\epsilon}{n_B}\right)_0 - m$$

Note: $g_\sigma \sigma$ has an upper limit ($m^* \neq 0$). $g_\omega \omega_0$ grows with ρ.

Isospin Force:

To distinguish n and p- interaction with ρ-meson exchange is introduced. Interaction part of this Lagrangian:

$$\mathcal{L}_{int} = -g_\rho \vec{\rho}_\nu . \vec{I}^\nu$$

where the vector (in isospin space) meson current:

$$\vec{I}^\nu = \frac{1}{2}\overline{\psi}\gamma^\nu \vec{\tau}\psi + \vec{\rho}_\mu \times \vec{\rho}^{\nu\mu} + 2g_\rho(\vec{\rho}^\nu \times \vec{\rho}^\mu) \times \vec{\rho}_\mu$$

Then in the EL-equation, the extra term is

$$\frac{\partial \mathcal{L}_{int}}{\partial \overline{\psi}} = \frac{g_\rho}{2}\gamma_\nu \vec{\rho}^\nu . \vec{\tau}\psi$$

Dirac eqn. becomes:

$$\left[\gamma_\mu \left(k^\mu - g_\omega \omega^\mu - \frac{1}{2} g_\rho \tau_3 \rho_3^\mu\right) - m^*\right]\psi(k) = 0$$

Other new equations:

As usual

$$g_\rho \rho_3^k = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 < \overline{\psi}\gamma^k \tau_3 \psi >= 0$$

$$g_\rho \rho_3^0 = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 < \overline{\psi}\gamma^0 \tau_3 \psi >= \left(\frac{g_\rho}{m_\rho}\right)^2 \frac{1}{2}(\rho_p - \rho_n)$$

Here $\pm 1/2$ are the isospin eigen values for p and n. In this case also three vector part ρ_3^k does not contribute because of same reason. Further, ρ_1 and ρ_2, which can be expressed in terms of ρ^+ and ρ^- do not contribute for obvious reason.

Energy density:

Energy eigen value:

$$\varepsilon_{I_3}(k) = E(k) + g_\omega \omega^0 + g_\rho I_3 \rho_3^0$$

where

$$E(k) = (k^2 + m^{*2})^{1/2}$$

Since $I_3|p>= +\frac{1}{2}|p>$ and $I_3|n>= -\frac{1}{2}|n>$, we have energy density

$$\begin{aligned}
\epsilon &= \frac{1}{3}bm(g_\sigma\sigma)^3 + \frac{1}{4}c(g_\sigma\sigma)^4 + \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\omega\omega_0^2 + \frac{1}{2}m_\rho\rho_{0_3}^2 \\
&+ \frac{1}{\pi^2}\int_0^{k_p} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega\omega_0 + \frac{1}{2}g_\rho\rho_3^0\right] \\
&+ \frac{1}{\pi^2}\int_0^{k_n} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega\omega_0 - \frac{1}{2}g_\rho\rho_3^0\right]
\end{aligned}$$

Pressure:

$$
\begin{aligned}
P &= -\frac{1}{3}bm(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 - \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\omega\omega_0^2 + \frac{1}{2}m_\rho\rho_{0_3}^2 \\
&+ \frac{1}{3}\frac{1}{\pi^2}\int_0^{k_p} k^2dk\frac{k^2}{(k^2+m^{*2})^{1/2}} \\
&+ \frac{1}{3}\frac{1}{\pi^2}\int_0^{k_n} k^2dk\frac{k^2}{(k^2+m^{*2})^{1/2}}
\end{aligned}
$$

Interacting Neutron Star Matter in Presence of Strong Magnetic Field

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Mean field interaction: Hartree and Hartree-Fock Interaction:

$\sigma - \omega - \rho$- meson exchange.

In a uniform magnetic field B along z-axis, the relativistic Hartree Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left[i\gamma_\mu D^\mu - m + g_\sigma\sigma - g_\omega\gamma_\mu\omega^\mu - \frac{1}{2}g_\rho\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\rho}^\mu\right]\psi \\ &+ \frac{1}{2}(\partial^\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 - \sum_{k=\omega,\rho}\left[\frac{1}{4}\left(\partial_\mu V_\nu^k - \partial_\nu V_\mu^k\right)^2 - \frac{1}{2}m_k^2(V_\mu^k)^2\right],\end{aligned}$$

Here, $D^\mu = \partial^\mu + iqA^\mu$, where the choice of gauge corresponding to the constant B along z-axis is $A_0 = 0$, $\mathbf{A} \equiv (0, xB, 0)$.

The general solution for protons is

$$\psi(\mathbf{r}) \propto \mathrm{e}^{-i\epsilon^H t + ip_y y + ip_z z} f_{p_y,p_z}(x),$$

where $f_{p_y,p_z}(x)$ is the 4-component spinor solution.

The Dirac-Hartree equation for protons in a magnetic field is then given by

$$\left[-i\alpha_x\partial/\partial x + \alpha_y(p_y - qBx) + \alpha_z p_z + \beta m^* + U_{0,p}^H\right] f_{p_y,p_z}^{(r)}(x) = \epsilon^H f_{p_y,p_z}^{(r)}(x)\ .$$

Equation of motion for neutrons $\longrightarrow q = 0$ and $U^H_{0;p} \longrightarrow U^H_{0;n}$; $\Longrightarrow$ plane wave soln.

For $T = 0$, only positive energy spinors: (Chiral representation)

$$f^{(1)}_{p_y,p_z}(x) = N_\nu \begin{pmatrix} (\epsilon^H_\nu + p_z) I_{\nu;p_y}(x) \\ -i\sqrt{2\nu q B_m} I_{\nu-1;p_y}(x) \\ -m^* I_{\nu;p_y}(x) \\ 0 \end{pmatrix},$$

$$f^{(2)}_{p_y,p_z}(x) = N_\nu \begin{pmatrix} 0 \\ -m^* I_{\nu-1;p_y}(x) \\ -i\sqrt{2\nu q B_m} I_{\nu;p_y}(x) \\ (\epsilon^H_\nu + p_z) I_{\nu-1;p_y}(x) \end{pmatrix},$$

$N_\nu = 1/\sqrt{2\epsilon^H_\nu(\epsilon^H_\nu + p_z)}$, and $\epsilon^H_\nu = \epsilon^H - U^H_{0;p} = (p_z^2 + m^{*2} + 2\nu q B_m)^{1/2} \longrightarrow$ effective Hartree energy.

Effective nucleon mass $m^* = m + U^H_S$ and $U^H_S = -(g_\sigma/m_\sigma)^2 n_S$.

Scalar density: $n_S = n_S^{(n)} + n_S^{(p)}$, with

$$n_S^{(n)} = \frac{m^*}{2\pi^2}\left[\mu_n^* \mathcal{O}_n^{1/2} - m^{*2} \ln \left\{\frac{\mu_n^* + \mathcal{O}_n^{1/2}}{m^*}\right\}\right] ,$$

$$n_S^{(p)} = \frac{m^* q B_m}{2\pi^2} \sum_{\nu=0}^{\nu_{\max}^{(p)}} g_\nu \ln \left[\frac{\mu_p^* + \mathcal{O}_{p,\nu}^{1/2}}{(m^{*2} + 2\nu q B_m)^{1/2}}\right] ,$$

where $\mathcal{O}_n = \mu_n^{*2} - m^{*2}$, and $\mathcal{O}_{p,\nu} = \mu_p^{*2} - m^{*2} - 2\nu q B_m$.

Interaction energy density: U_0^H (protons and neutrons) $U_{0;p}^H = (g_\omega/m_\omega)^2 n_B + (g_\rho/m_\rho)^2 \rho_3/4$ and $U_{0;n}^H = (g_\omega/m_\omega)^2 n_B - (g_\rho/m_\rho)^2 \rho_3/4$,

where $\rho_3 = n_p - n_n$.

Total baryon number density: $n_B = n_n + n_p$, with

$$n_n = \frac{\mathcal{O}_n^{3/2}}{3\pi^2} , \quad n_p = \frac{q B_m}{2\pi^2} \sum_{\nu=0}^{\nu_{\max}^{(p)}} g_\nu \mathcal{O}_{p,\nu}^{1/2} .$$

The total energy density:

$$
\begin{aligned}
\epsilon &= \frac{g_\sigma^2}{2m_\sigma^2} n_S^2 + \frac{g_\omega^2}{2m_\omega^2} n_B^2 + \frac{g_\rho^2}{8m_\rho^2} \rho_3^2 \\
&+ \frac{1}{8\pi^2}\left[2\mu_n^{*3}\mathcal{O}_n^{1/2} - m^{*2}\mu_n^*\mathcal{O}_n^{1/2} - m^{*4}\ \mathrm{ln}\ \left\{\frac{\mu_n^* + \mathcal{O}_n^{1/2}}{m^*}\right\}\right] \\
&+ \frac{qB_m}{4\pi^2}\sum_{\nu=0}^{\nu_{\mathrm{max}}^{(p)}} g_\nu \left[\mu_p^*\mathcal{O}_{p,\nu}^{1/2} + m_{p,\nu}^{*2}\ \mathrm{ln}\ \left\{\frac{\mu_p^* + \mathcal{O}_{p,\nu}^{1/2}}{m_{p,\nu}^*}\right\}\right] \\
&+ \frac{qB_m}{4\pi^2}\sum_{\nu=0}^{\nu_{\mathrm{max}}^{(e)}} g_\nu \left[\mu_e\mathcal{O}_{e,\nu}^{1/2} + m_{e,\nu}^{2}\ \mathrm{ln}\ \left\{\frac{\mu_e + \mathcal{O}_{e,\nu}^{1/2}}{m_{e,\nu}}\right\}\right] .
\end{aligned}
$$

Here $\mathcal{O}_{e,\nu} = \mu_e^2 - m_e^2 - 2\nu qB_m$ and $m_{i,\nu}^{*2} = m_i^{*2} + 2q\nu B_m$, where m_i^*s denote m^*s(m_es) for $i = p(e)$.

The total pressure: $P = n_B^2 \partial(E/A)/\partial n_B$, where E/A is the energy per baryon.

Hartree-Fock ($\sigma-\omega$-meson exchange)

B_m is along z-axis, Lagrangian:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left[i\gamma_\mu D^\mu - m - g_\sigma\sigma - g_\omega\gamma_\mu\omega^\mu\right]\psi + \frac{1}{2}(\partial^\mu\sigma)^2 \\ &- \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{4}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu\right)^2 + \frac{1}{2}m_\omega^2(\omega_\mu)^2,\end{aligned}$$

Solution for protons:

$$\psi(\mathbf{r}) \propto \exp(-i\epsilon^{HF}t + ip_y y + ip_z z) f_{p_y,p_z}(x),$$

We take $\nu = 0$, then

$$f_{p_y,p_z}^{\nu=0}(x) = N_{\nu=0}\begin{pmatrix}\epsilon_{\nu=0}^{HF} + p_{vz} \\ 0 \\ -m^* \\ 0\end{pmatrix} I_{\nu=0;p_y}(x),$$

where $N_{\nu=0} = 1/\sqrt{2\epsilon_{\nu=0}^{HF}(\epsilon_{\nu=0}^{HF} + p_{vz})}$ and $\epsilon_{\nu=0}^{HF} = \epsilon_{p_z}^{HF} - U_0^H - U_0^F(p_z) = \sqrt{p_{vz}^2 + m^{*2}}$.

The DHF equation of protons for $\nu = 0$:

$$\left[\alpha_z p_z + \beta\left(m + U^H + U^F\right)\right]u(p_z) = \epsilon_{p_z}^{HF}u(p_z),$$

Effective mass $m^* = m + U_S^H + U_S^F(p_z)$, and $u(p_z)$ is the momentum dependent part of the spinor.

Hence one can obtain the EOS etc of interacting neutron star matter in Hartree-Fock approximation.

Tolmann-Oppenheimer-Volkoff (TOV) Equation for Neutron Stars:

We use Gravitational or Geometrical units:
$G = c = 1$.

Hence the TOV or the GR hydrostatic equilibrium equation along with the subsidiary mass equation:

$$\begin{aligned} \frac{dP}{dr} &= -\frac{\rho m}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi P r^2}{m}\right)\left(1 - \frac{2m}{r}\right)^{-1} \\ \frac{dm}{dr} &= 4\pi r^2 \rho \end{aligned}$$

TOV equation is obtained from GR Einstein's equation with Schwarzschild metric, valid for a static, non-rotating system in vacuum.

How to solve the equations (numerically)?:

1. EOS $P(\rho)$ is known from the core to the crust.
2. Pick a value of central density ρ_c. The Pressure is known.
3. At the centre $m = 0$ (take an extremely small number for numerical calculation.)
4. Integrate the above equations out ward from $r = 0$.
5. Each time a new value for ρ and also a new value for $m(r)$ will be obtained, hence get $P(\rho)$.
6. At $r = R$, the radius of the star, $P = 0$.
7. At $r =$, $m(R) = M$, the mass of the star.
8. Hence we get $M(R)$ and density profile for a given ρ_c
9. Change the value of ρ_c and repeat (1-8).
10. We get $M(\rho_c)$.
11. The value of ρ_c be such that $dM/d\rho_c > 0$, otherwise the system becomes general relativistically unstable.

At the core of NS chemical equilibrium among the constituents: $\Longrightarrow$

$n \rightarrow p + e^- + \overline{\nu}_e$, $p + e^- \rightarrow n + \nu_e \Longrightarrow \mu_n = \mu_p + \mu_e$. Neutrinos are non-degenerate, leave the immediately after their formation.

Charge neutrality: $n_p = n_e$.

Self-consistent solution of these equations along with the equations discussed in $\sigma - \omega - \rho$-meson model will give EOS for the core material.

More complicated cases:

(i) If $\mu_e > m_\mu$, μ-mesons or muons will be created.
(ii) if $\mu_{n-p} > m_B$, B-is some baryon resonances- they have to be considered.
(iii) Presence of π-mesons and kaons are also important.

Schwarzschild exterior line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$M \rightarrow$ total mass. The interior line element is given by

$$ds^2 = e^{2\phi} dt^2 - e^{2\lambda} dr^2 + r^2 d\Omega^2$$

where ϕ and λ are functions of r and t and $d\Omega^2 \equiv d\theta^2 = \sin^2\theta d\phi^2$.

GR Hydro-static Stability Equation:

Spherically symmetric non-rotating neutron star- obtained from the Einstein equation:

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}\right)$$

where

$$g_{\mu\nu} = \begin{pmatrix} (1 - 2M/r) & 0 & 0 & 0 \\ 0 & -(1 - 2M/r)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

The energy momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = -Pg_{\mu\nu} + (P+\rho)u_\mu u_\nu$$

$u_\mu u^\mu = 1$, $P \rightarrow$ kinetic pressure and $\rho \rightarrow$ matter density.

Components of Ricci tensor $R_{\mu\nu}$- Schwarzschild metric- from the Einstein's equation $\Rightarrow$

$$\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2 \rho(r) \\
\frac{dP}{dr} &= -\frac{\rho(r)P(r)}{r^2}\left(1+\frac{P(r)}{\rho(r)}\right),\left(1+\frac{4\pi P(r)r^2}{m(r)}\right)\left(1-\frac{2m(r)}{r}\right)^{-1}, \\
\frac{d\phi}{dr} &= -\frac{1}{\rho(r)}\frac{dP}{dr}\left(1+\frac{P(r)}{\rho(r)}\right)^{-1}
\end{aligned}$$

The Oppenheimer-Volkoff or Tolman, Oppenheimer-Volkoff equation (OV or TOV equation) .

$G = c = 1$.

complementary mass equation -$m(r)$, *mass inside radius* r

$\phi \rightarrow$ gravitational potential in the Newtonian limit.

Numerical soln. of TOV equation along with the supplementary mass equation- follow the following steps:

a) Choose a central density ρ_c. From the equation state obtain P_c, the central pressure. Further $m(r = 0) = 0$.

b) Integrate the eqns numerically from the centre ($r = 0$) to the surface. During integration, each time a new value of P and from the equation of state ρ are obtained.

c) Since pressure vanishes at the star's surface, the value $r = R$, the radius of the star is obtained for $P = 0$, and the corresponding $m(r = R) = M$, the mass of the star.

$m(R)$ must equal $M \Leftarrow$ interior metric coefficient:

$$e^{2\lambda} \equiv \left(1 - \frac{2m}{r}\right)^{-1}$$

match smoothly to the exterior Schwarzschild metric coefficient:

$$e^{2\lambda} \equiv \left(1 - \frac{2M}{R}\right)^{-1}$$

Analytical Soln. of TOV Eqn.- uniform matter density approximation, i.e., ρ =constant. $\Rightarrow$

$$M = M(R) = \frac{4\pi}{3}\rho R^3$$

and

$$P(r) = \frac{3M}{4\pi R^2} \frac{\left(1 - \frac{2M}{R}\right)^{1/2} - \left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}}{\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2} - 3\left(1 - \frac{2M}{R}\right)^{1/2}}$$

$P(r = R) = 0 \Rightarrow$ surface.

Important feature of the soln.:

Constraint connecting the star's mass and radius- pressure at the centre (say at $r = r_0$) can be infinity provided

$$r_0^2 = 9R^2 - \frac{4R^3}{M}$$

Avoid this unphysical result $\Rightarrow$ r_0 - imaginary, i.e.,

$$\frac{M}{R} < \frac{4}{9}$$

true for any equation of state. The same constraint can also be obtained directly from the Einstein field equation.

Physical meaning: Pack more mass- fixed radius or contract- fixed mass $\Rightarrow$ above inequality breaks $\Rightarrow$ destroy the hydrostatic equilibrium condition (due to increased gravitational attraction).

The above constraint can be re-written as

$$M^2 < \frac{16}{243\pi\rho}$$

Geometrical Units

Choice of unit: $c = G = 1 \longrightarrow$ time is in cm: 1sec$= 3 \times 10^{10}$cm, mass is in length unit: 1g$= 0.7425 \times 10^{-28}$cm. Then $M_\odot = 1.4766$km.

Something interesting happening at the radii $r = 0 \longrightarrow$ the real singularity and $r = R_s = 2M \longrightarrow$ the Schwarzschild radius- Schwarzschild singularity- characteristic length scale for curvature in the Schwarzschild geometry.

Newtonian Neutron Stars with Polytropic Equation of State

Newtonian limit- rest mass density $\rho_0(r)$ dominates over energy density $\rho(r)$- gravitational potential $2GM/r$ is everywhere small enough $\longrightarrow$ TOV equation $\Rightarrow$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_0}\frac{dP}{dr}\right) = -4\pi G\rho_0$$

Polytropic EOS: $P = K\rho_0^\Gamma$ - adiabatic EOS, K and Γ - constants.

With $\Gamma = 1 + 1/n$, n - polytropic index, writing

$$\begin{aligned} \rho_0 &= \rho_c\theta^n \\ r &= a\xi \\ a &= \left[\frac{(n+1)K\rho_c^{(1/n-1)}}{4\pi G}\right]^{1/2} \end{aligned}$$

$\rho_c = \rho_0(r = 0)$ - central density $\Rightarrow$ hydrostatic stability equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

-Lane-Emden equation -second order differential equation- unique solution- two boundary conditions: $\theta(0) = 1$ and $\theta'(0) = 0$

Integrated numerically- starting from $\xi = 0$- for $n < 5$ ($\Gamma > 6/5$)- solution decreases monotonically and $\rightarrow 0$ at a finite value, say $\xi = \xi_1$- stars surface- $\rho_0 = P = 0 \Rightarrow$ the radius of the star:

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2} \xi_1$$

and the mass:

$$M = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(3-n)/2} \xi_1^2 \mid \theta'(\xi_1) \mid$$

Eliminating $\rho_c \Rightarrow$ mass-radius relation:

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_1^{(3-n)/(1-n)} \xi_1^2 \mid \theta'(\xi_1) \mid$$

Special solutions: low density non-relativistic

1. $\Gamma = \frac{5}{3},\ n = \frac{3}{2},\ \xi_1 = 3.65375,\ \xi_1^2 \mid \theta'(\xi_1) \mid = 2.71406,$

   high density ultra – relativistic

2. $\Gamma = \frac{4}{3},\ n = 3,\ \xi_1 = 6.89685,\ \xi_1^2 \mid \theta'(\xi_1) \mid = 2.01824$

Hence for white dwarfs: NR

$$R \approx 1.22 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ gm cm}^{-3}}\right)^{-1/6} \text{ km}$$

$$M \approx 0.7011 \left(\frac{R}{10^4 \text{ km}}\right)^{-3} M_\odot$$

Relativistic Case:

$$R \approx 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ gm cm}^{-3}}\right)^{-1/3} \text{ km}$$

$$M \approx 1.457 M_\odot$$

This limiting value of White Dwarf mass- The Chandrasekhar mass.